Evolution of the population of *Microtus Epiroticus*: the Yoccoz-Birkeland model.

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Abstract

We study the discretized version of a dynamical system given by a model proposed by Yoccoz and Birkeland to describe the evolution of the population of *Microtus Epiroticus* on Svalbard Islands, see http://zipcodezoo.com/Animals/M/Microtus_epiroticus. We prove that this discretized version has an attractor Λ with a hyperbolic 2-periodic point p in it. For certain values of the parameters the system restricted to the attractor exhibits sensibility to initial conditions. Under certain assumptions that seems to be sustained by numerical simulations, the system is topologically mixing (see definition 4.1) explaining some of the high oscillations observed in Nature. Moreover, we estimate its order-2 Kolmogorov entropy obtaining a positive value. Finally we give numerical evidence that there is a homoclinic point associated with p.

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Contents

1 Introduction

We study the evolution of the population of *Microtus Epiroticus* (sibling vole) on Svalbard Islands in the Arctic Ocean, using a model proposed by J. C. Yoccoz and H. Birkeland, see [Ar]. It is known that there are no significant predation of these small mammals but in spite of that, the population presents high oscillations in its number albeit the lack of food is not a determinant factor to the occurrence of these phenomena. This population exhibits dramatic multi-annual fluctuations, by a factor greater than 20, [YI].

The Sibling Vole (*Microtus Epiroticus*) is a species of vole found through much of northern Europe. First discovered in 1960 in the Grumantbyen area, they were thought to be the Common Vole until a genetic analysis correctly identified them in 1990, [FJASY].

Since these rodents were introduced from Russia on Svalbard Isles between 1930 and 1960, [YI], the annual oscillations of their number may be explained, at least in part, by a non total adaptation to the environment, and by the pronounced seasonal fluctuation in climatic variability at Svalbard where temperatures of -30 degrees Celsius are common, see [YI, LBY].

Figure 1: Microtus Epiroticus.

Let us first sketch the taxonomy classification of *Microtus Epiroticus*.
- Domain: Eukaryota
-   Kingdom: Animalia
-     Phylum: Chordata
-      Class: Mammalia
-       Order: Rodentia
-        Family: Muridae
-         Subfamily: Arvicolinae
-          Genus: Microtus
-           Species: Microtus Epiroticus

Jean Christophe Yoccoz and H. Birkeland, see [Ar], have proposed the following equation

$$N(t) = \int_{A_0}^{A_1} N(t-a)m(N(t-a))m_\rho(t-a)S(a)da \,, \qquad (1)$$

to model the evolution in time of the population of *Microtus Epiroticus*. In the equation it is taken into account only the number $N(t)$ of fertile females at certain time t. Indeed, the inclusion in a model of the number of males is justified when there are difficulties for a female to find a male (for instance if the density of population is too small or if the ratio male-population : female-population is far away from $1:1$) which is not the case for these rodents. In fact as has been pointed out by R. A. Ims in [Ims], "spatial clumping of sexually receptive females induces space sharing among male voles" which implies that it is not difficult for a female to find a male. Moreover, the quantity of females is about the same as those of males for these rodents, [Ims2, YI].

Let us describe the parameters of the model given by equation (1):

1. t: is the time measured in years.
2. $N(t)$: is the population of active females at time t.
3. A_0: is the maturation age,
4. A_1: is the maximal age expected for *Microtus Epiroticus*.
5. $m(N)$: annual individual reproduction rate for a population of N individuals
6. $m_\rho(t)$: is the reproduction probability at time t of the year.
7. $S(a)$: probability to survive up to a years.

The model take into account the following facts:

(a) The age when the females of *Microtus Epiroticus* have their first offspring is about 50 days, i.e., $A_0 \approx 0.14$ years (see [YIS]).

(b) The maximal age of survival is about 2 years, i.e., $A_1 = 2$ (see [YI]).

(c) The seasonal factor $m_\rho(t)$, that is, the reproduction probability at time t of the year, varies sharply from 0 in Winter to 1 from Spring to Autumn. Thus, the definition of $m_\rho(t)$ we adopt is

$$m_\rho(t) = \begin{cases} 0 & \text{if } 0 \leq t \mod (1) < \rho \\ 1 & \text{if } \rho \leq t \mod (1) < 1 \end{cases}.$$

(d) The annual individual reproduction rate $m(N)$ for a population of N individuals, is too high when $N(t)$ is small. Indeed $m(N)$ of the order of a constant $m_0 > 30$ individuals is realistic due to the high fertility of these rodents. The value of $m(N(t))$ decays sharply when the population $N(t)$ increases. Following [Ar], for $m(N)$ we adopt

$$m(N) = \begin{cases} m_0 & \text{if } N \leq 1 \\ m_0 N^{-\gamma} & \text{if } N > 1 \end{cases} ; \quad \gamma > 1. \tag{2}$$

We will assume that $\gamma > 1$ and for some calculations we take $\gamma = 8.25$. The reason for that is that there is numerical evidence, see [Ar], that for this value of the parameter we have chaotic behavior.

(e) Finally for the survival probability $S(a)$, again following [Ar], we consider a linear function:

$$S(a) = 1 - \frac{a}{A_1}, \quad \text{if} \quad 0 \leq a \leq A_1, \quad \text{and} \quad S(a) = 0 \quad \text{elsewhere}.$$

Remark 1.1. *Another choice of functions for $S(a)$, for instance $S(a) = \exp(-\kappa a)$, with $\kappa > 0$, are also usual in the literature. It would be interesting to test the model given by (1) replacing the linear function at (e) by a exponential one.*

Let us describe how the integral equation

$$N(t) = \int_{A_0}^{A_1} N(t-a)m(N(t-a))m_\rho(t-a)S(a)da \quad \text{arises}.$$

For $N(t)$, the contribution of females of age in between $[a, a+\Delta a] \subset [A_0, A_1]$ is

$$\text{fem}(t-a) \times (\text{reprod. rate}(t-a) \times (\text{ season factor}(t-a) \times (\text{prob. survive}) \times \ell([a, a+\Delta a])$$

$$= N(t-a) \times m(N(t-a)) \times m_\rho(t-a) \times S(a) \times \Delta a,$$

where $\ell(J)$ is the length of the interval J and fem(t) is the number of females at time t. Here we assume that a female of age near A_1 can reproduce and Δa is small. Taking a partition $\{a_0 = t - A_1, a_1, \ldots, a_n = t - A_0\}$ of the interval $[t - A_1, t - A_0]$ we find

$$N(t) \approx \sum_{j=0}^{n-1} N(t-a_j)m(N(t-a_j))m_\rho(t-a_j)S(a_j)\Delta a_j, \quad \text{where} \quad \Delta a_j = (a_{j+1} - a_j)$$

Letting $n \to \infty$ we get at the limit the integral equation given by (1).

1.1 The discrete model.

There is no special reason to prefer the continuous model above to its discretization: most of the quantities involved, as $N(t)$ and $m(N)$, are by nature of discrete type. Moreover, from the experimental point of view, it is more natural to split the year on days and even in groups of days since it is very difficult to monitor $N(t)$ experimentally. Hence, we assume that the year is split into p equal parts.

Since the expected value of survival is bounded by $A_1 = 2$ years we will study the evolution of $N(t)$ for discrete values of t, modeling the period $[0, A_1]$ as a vector of $A_1 p + 1$ real entrances, from $t = 0$ at the initial time of the first year, to $t = 2p$ corresponding to $A_1 = 2$ the final time of the second year.

In this case the probability of survival at age $\frac{j}{p}$ is given by

$$S(j) = 1 - \frac{j}{2p}, \quad j = 0, 1, 2 \dots, 2p.$$

where $A_1 p = 2p$. It is also convenient to consider $S(j) = 1 - \frac{j}{2p+1}$. This takes into account the case where $S(2p) > 0$, i.e., when these animals can reproduce till the final of their lives.

Given an initial vector value $(N_0, N_1, N_2, \dots, N_{A_1 p-1}, N_{A_1 p}) \in {I\!\!R}^{A_1 p+1}$, the evolution of $N(t) = N_t$, $t \in {I\!\!N}$, is governed by

$$N_t = \sum_{h=A_0 p}^{A_1 p-1} N_{t-h} m(N_{t-h}) m_\rho(t-h) S(h) \Delta h \tag{3}$$

$$= \frac{1}{p} \sum_{h=A_0 p}^{2p-1} N_{t-h} m(N_{t-h}) m_\rho(t-h) S(h).$$

Next we explain the choices in equation (3).

1. We take $A_0 p = \left[\frac{50 \times p}{365}\right] \approx 14$ which corresponds to the age at which the females have their first litter of pups (about 50 days). Note that if $p = 100$ then $A_0 = 0.14$ corresponds to 51 days.

2. We take $\Delta h = \frac{1}{p}$ years that corresponds to the length of the unit interval in which we split the year. When $p = 100$ this gives $\Delta h = \frac{1}{100}$ years = 3.65 days.

Note that the value of N at t depends only on the values of N in $[t - A_1; t - A_0]$. Thus, the knowledge of N_t for $t \in [-A_1 p, 0]$ (two years of observation) enables us to predict N_t for $t \in [0, A_0 p]$. When $p = 100$ and $A_0 = 0.14$ this means that the knowledge of $(N_0, N_1, N_2, \dots, N_{200})$ enables us to compute $N_{201}, \dots, N_{214}$. Recursively we may compute N_j for all $j \geq 0$.

2 The dynamical system

Equation (3) defines a discrete dynamical system in ${I\!\!R}^{2p+1}$ as follows:

$$(N_0, N_1, \dots, N_{2p}) \mapsto T(N_0, N_1, \dots, N_{2p}) = (N_p, N_{p+1}, \dots, N_{3p}),$$

where we have used that $A_1 = 2$ and $T : {I\!\!R}^{2p+1} \to {I\!\!R}^{2p+1}$ is defined recursively by equation (3) for $t = 2p+1, \dots, 3p$.

In order to describe theoretical properties of a system given by the discretized version (3) of Yoccoz-Birkeland equation (1), let us assume the following restrictions that weaken those given by conditions (a)–(e) described before. Doing this allows to apply the conclusions to different species respecting equation (3) and those restrictions. In particular, these conclusions will apply to the original system modeling *Microtus Epiroticus*.

1. $m(N)$ is a continuous function, $m : {I\!\!R}^+ \to {I\!\!R}^+$,

2. there is $m_0 \in {I\!\!R}^+$ such that

$$\begin{cases} m_0 \geq m(N) \geq m_0/2 & \text{if} \quad N \leq 1 \quad \text{and} \\ m_0 N^{-\gamma} \geq m(N) \geq \min\{\frac{m_0}{2}, m_0 \cdot N^{-\gamma}\} & \text{if} \quad N > 1, \end{cases} \tag{4}$$

3. $0 \leq m_\rho(t) \leq 1$ (so that we now allow $0 < m_\rho(t) < 1$ for certain values of t),

4. There is $\epsilon \geq 0$ such that $m_\rho(t) = 1$ for t in an interval of length $1 - \rho - \epsilon > 0$,

5. $0 < 2A_0 < A_1$ and $A_0 + 1 < A_1$ (this means that in average each individual has at least two opportunities to reproduce),

6. Defining c_0 as $c_0 = \frac{1}{p}\left(\sum_{h=A_0p+(\rho+\epsilon)p}^{A_0p+p} S(h)\right)$ we require $c_0\, m_0 > 2$. From the definition of $S(h)$ it follows

$$c_0 = \frac{1}{p}\left(\sum_{h=A_0p+(\rho+\epsilon)p}^{A_0p+p}(1-\frac{h}{pA_1})\right) = (1-\rho-\epsilon)\left(1-\left(\frac{(1+\rho+\epsilon)+2A_0}{2A_1}\right)\right). \qquad (5)$$

The condition $c_0\, m_0 > 2$ will imply, as we will see below in Proposition 2.3, that the population does not extinguish, that is, it has the permanence property (see Definition 2.1).

7. The exponent γ satisfies $\gamma > 1$.

The following proposition shows that a dynamical system governed by equation (3) and respecting the restrictions 1. to 7. above is bounded.

Proposition 2.1. *For all $t = 1, 2, \ldots, A_0p$ we have $N_t \leq N_{max} := m_0\left(\frac{(A_1-A_0)^2}{2A_1}\right)$.*

Proof. Since $\gamma > 1$, inequalities (4) imply $N_j m(N_j) \leq m_0$ for all j. Moreover from $m_\rho \leq 1$ we obtain

$$N_t = \sum_{h=A_0p}^{A_1p-1} N(t-h)m(N(t-h))m_\rho(t-h)S(h)\Delta h \leq$$

$$\sum_{h=A_0p}^{A_1p} m_0 S(h)\frac{1}{p} = \frac{m_0}{p}\sum_{h=A_0p}^{A_1p} S(h) = \frac{m_0}{p}\sum_{h=A_0p}^{A_1p}(1-\frac{h}{pA_1}) =$$

$$\frac{m_0}{p}\left((A_1-A_0)p - \frac{A_1p(A_1p+1)-A_0p(A_0p+1)}{2A_1p}\right) \leq$$

$$m_0\left((A_1-A_0) - \frac{A_1(A_1+1/p)-A_0(A_0+1/p)}{2A_1}\right) \leq$$

$$m_0\left((A_1-A_0) - \frac{A_1^2-A_0^2}{2A_1}\right) = m_0\left(\frac{(A_1-A_0)^2}{2A_1}\right).$$

$\square$

By induction we obtain that for all $t \geq 0$, $N_t \leq N_{max}$.

Remark 2.2. *For the values $A_1 = 2$, $A_0 = 0.18$, $m_0 = 50$ we have $N_{max} \approx 41.4$.*

2.1 Permanence.

In this section we verify that the population given by equation (3) and respecting the restrictions 1. to 7. above, in particular conditions (4) and (5), does not extinguish.

Definition 2.1. *We say that a system $P(t)$ modeling the evolution of a population is* permanent, *or satisfies the* permanence *property, if for any positive initial vector value P_0, there is $\epsilon > 0$ such that the solution $P(t)$ satisfies*

$$\liminf_{t\geq 0} P(t) \geq \epsilon.$$

If a given system is permanent then, assuming that the environmental conditions do not change in time, the associated population will not extinguish. Thus, concerning with population dynamics this property is very important.

The next proposition shows that the system under study satisfies the permanence property.

Proposition 2.3. *If $i(N) = \min\{N_t, t \in [-pA_1, 0]\} > 0$ then $N_t > 0$ for all $t = 0, 1, \ldots, A_0p$. Moreover,*

- *If $i(N) \leq N_{max}^{1-\gamma}$ then $N_t \geq \frac{c_0\, m_0}{2}i(N) > i(N)$, $t \in [0, pA_0]$.*
- *If $i(N) \geq N_{max}^{1-\gamma}$ then $N_t \geq \frac{c_0\, m_0}{2}N_{max}^{1-\gamma}$, $t \in [0, pA_0]$.*

Proof. If $N(t - h) \leq 1$ then, by (4),

$$N(t - h)m(N(t - h)) \geq N(t - h)\frac{m_0}{2} \geq i(N)\frac{m_0}{2}.$$

Otherwise $N(t - h) > 1$ and then, again by (4),

$$N(t - h)m(N(t - h)) \geq \min\{N(t - h)^{1-\gamma}\frac{m_0}{2}, N(t - h)\frac{m_0}{2}\} \geq \min\{N_{max}^{1-\gamma}\frac{m_0}{2}, i(N)\frac{m_0}{2}\}.$$

Hence we have

$$N_t = \sum_{h=A_0 p}^{2p-1} N(t - h)m(N(t - h))m_\rho(t - h)S(h)\Delta h \geq$$

$$\frac{1}{p}\min\{N_{max}^{1-\gamma}\frac{m_0}{2}, i(N)\frac{m_0}{2}\} \sum_{h=A_0 p}^{2p-1} m_\rho(t - h)S(h) \geq$$

$$\min\left\{N_{max}^{1-\gamma}\frac{m_0}{2}, i(N)\frac{m_0}{2}\right\} \frac{1}{p}\left(\sum_{h=A_0 p+(\rho+\epsilon)p}^{A_0 p+p}(1 - \frac{h}{pA_1})\right) = \min\left\{N_{max}^{1-\gamma}\frac{c_0 m_0}{2}, i(N)\frac{c_0 m_0}{2}\right\}.$$

Since, by (5), $c_0 m_0 > 2$ we get by induction that $N(t) > 0$ for all $t \in [0, A_0 p]$. $\qquad\square$

Clearly Proposition 2.3 implies that $N_t > 0$ for all $t \geq 0$.

Corollary 2.4. *There is $t_0 > 0$, depending on the initial vector value, such that we have $N(t) \geq \frac{c_0 m_0}{2}N_{max}^{1-\gamma}$, $t \geq t_0$.*

2.2 Existence of fixed points.

The following corollary is a straightforward consequence of Propositions 2.1 and 2.3.

Corollary 2.5. *If $N_t > 0$ for all $t \in [-A_p, 0]$ then there is $t_0 > 0$ such that $\frac{c_0 m_0}{2}N_{max}^{1-\gamma} \leq N_t \leq N_{max}$ for $t \geq t_0$. In particular T maps the compact set*

$$\mathcal{K} = \left[\frac{c_0 m_0}{2}N_{max}^{1-\gamma}, N_{max}\right]^{pA_1+1} \qquad \text{into itself.}$$

$\qquad\square$

Now set
$$H := \left\{N = (N_0, N_1, \ldots, N_{2p}) \in I\!\!R^{2p+1} : \forall j = 0, 1, \ldots 2p : N_j > 0\right\}. \tag{6}$$

Observe that Proposition 2.3 together with Corollary 2.4 imply that T maps H into itself.

Next we prove that $T : H \to H$ is Lipschitz.

Lemma 2.6. *$T : H \to H$ is a Lipschitz function.*

Proof. We put in $I\!\!R^{2p+1}$ the sup norm: $\|x\| = \|(x_0, x_1, \ldots, x_{2p})\| = \sup_{t=0,\ldots,2p} |x_j|$.

From the definition of T we have $T(N_0, N_1, \ldots, N_{2p}) = (N_p, N_{p+1}, \ldots, N_{3p})$. Hence for all $j = 0, \ldots, p$ we have

$$|(T(N) - T(N'))_j| = |N_{j+p} - N'_{j+p}| \leq \|N - N'\|. \tag{7}$$

For $j = p + 1, \ldots, 2p$, the difference $|N_{(t-h)}m(N_{(t-h)}) - N'_{(t-h)}m(N'_{(t-h)})|$ can be estimated as follows:

(a) If $N_{(t-h)} \leq 1$ and $N'_{(t-h)} \leq 1$ then by inequalities (4) we have that

$$|N_{(t-h)}m(N_{(t-h)}) - N'_{(t-h)}m(N'_{(t-h)})| \leq m_0|N_{(t-h)} - N'_{(t-h)}|.$$

(b) If $N_{(t-h)} \geq 1$ and $N'_{(t-h)} \geq 1$ then, again by (4), we have that

$$|N_{(t-h)}m(N_{(t-h)}) - N'_{(t-h)}m(N'_{(t-h)})| \leq |(N_{(t-h)})^{1-\gamma} - (N'_{(t-h)})^{1-\gamma}|m_0.$$

By the Mean Value Theorem, there is $\widetilde{N} \in (N_{(t-h)}, N'_{(t-h)})$ such that

$$|(N_{(t-h)})^{1-\gamma} - (N'_{(t-h)})^{1-\gamma}| = |1 - \gamma|\widetilde{N}^{-\gamma}|N_{(t-h)} - N'_{(t-h)}|.$$

Since $\gamma > 1$ and $\widetilde{N} > 1$ we obtain

$$m_0|(N_{(t-h)})^{1-\gamma} - (N'_{(t-h)})^{1-\gamma}| \leq m_0(\gamma - 1)|N_{(t-h)} - N'_{(t-h)}|.$$

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(c) If one of the above quantities is greater than 1 and the other is not, say $N'_{(t-h)} > 1$ and $N_{(t-h)} \leq 1$, then

$$|N_{(t-h)}m(N_{(t-h)}) - N'_{(t-h)}m(N'_{(t-h)})| = m_0|N_{(t-h)} - (N'_{(t-h)})^{1-\gamma}|.$$

If $N_{(t-h)} \geq (N'_{(t-h)})^{1-\gamma}$ then, since $0 < N_{(t-h)} \leq 1$ and $1 - \gamma < 0$ we get

$$m_0|N_{(t-h)} - (N'_{(t-h)})^{1-\gamma}| = m_0(N_{(t-h)} - (N'_{(t-h)})^{1-\gamma}) \leq$$

$$m_0((N_{(t-h)})^{1-\gamma} - (N'_{(t-h)})^{1-\gamma}) =$$

$$m_0|(N_{(t-h)})^{1-\gamma} - (N'_{(t-h)})^{1-\gamma}| \leq m_0(\gamma - 1)|N_{(t-h)} - N'_{(t-h)}|.$$

Otherwise, if $N_{(t-h)} < (N'_{(t-h)})^{1-\gamma}$ then, since $N'_{(t-h)} > 1$ and $1 - \gamma < 0$, we have $0 > N_{(t-h)} - (N'_{(t-h)})^{1-\gamma} > N_{(t-h)} - N'_{(t-h)}$ and therefore

$$|N_{(t-h)}m(N_{(t-h)}) - N'_{(t-h)}m(N'_{(t-h)})| = m_0|N_{(t-h)} - (N'_{(t-h)})^{1-\gamma}| \leq m_0|N_{(t-h)} - N'_{(t-h)}|.$$

Next, to estimate $|N_t - N'_t|$ for $t = p, p+1, \ldots, A_0 p$, we use (a), (b) and (c) above as below. Let $L = \max\{m_0, m_0(\gamma - 1)\}$. Taking into account that $m_\rho(t - h)$ and $S(h)$ are between 0 and 1 and $\Delta h = \frac{1}{p}$ we obtain that:

$$|N_t - N'_t| = \left| \sum_{h=A_0 p}^{2p-1} N_{(t-h)}m(N_{(t-h)})m_\rho(t-h)S(h)\Delta h - \sum_{h=A_0 p}^{2p-1} N'_{(t-h)}m(N'_{(t-h)})m_\rho(t-h)S(h)\Delta h \right| =$$

$$\left| \frac{1}{p} \sum_{h=A_0 p}^{2p-1} \left(N_{(t-h)}m(N_{(t-h)}) - N'_{(t-h)}m(N'_{(t-h)}) \right) m_\rho(t-h)S(h) \right| \leq$$

$$\frac{1}{p} \sum_{h=A_0 p}^{2p-1} \left| N_{(t-h)}m(N_{(t-h)}) - N'_{(t-h)}m(N'_{(t-h)}) \right| m_\rho(t-h)S(h) \leq$$

$$\frac{1}{p} \sum_{h=A_0 p}^{2p-1} L|N_{(t-h)} - N'_{(t-h)}| \leq \frac{1}{p} \sum_{h=A_0 p}^{2p-1} L \max_h |N_{(t-h)} - N'_{(t-h)}| \leq (A_1 - A_0)L\|N - N'\|. \tag{8}$$

Taking into account that $1 \leq (A_1 - A_0)L$ and inequalities (7) and (8) we have that for all $j = 0, \ldots, p, p+1, \ldots, p + A_0 p$, $|T(N) - T(N')|_j \leq (A_1 - A_0)L \cdot \|N - N'\|$. By induction, since $A_0 p > 1$, we obtain that $\|T(N) - T(N')\| \leq (A_1 - A_0)L \cdot \|N - N'\|$ finishing the proof. $\square$

Corollary 2.7. *There is a fixed point p for $T : \mathcal{K} \to \mathcal{K}$.*

Proof. By Lemma 2.6 the map T is Lipschitz hence continuous. Moreover $\mathcal{K}$ is a $(2p+1)$-dimensional topological disk. Hence Brouwer Fixed Point Theorem applies, [Sp, Chapter 4, Section 7]. $\square$

Remark 2.8. *Since every two years $(A_1 = 2)$ the rodent population is renewed perhaps it is more natural to search for fixed points for $T^2 : \mathcal{K} \to \mathcal{K}$. So, we are interested in both, fixed points and period-two points $N \in \mathcal{K}$. Their existence is guaranteed by Corollary 2.7.*

In Appendix E we estimate the coordinates of a fixed point p of $T^2 : H \to H$. We find that the distance given by the norm of the supremum between p and $T^2(p)$ is about 8.0148×10^{-14} and the l^1 norm is about 4.0353×10^{-12}. This estimate of p is better than that obtained by Arlot, [Ar, Section B.8], which is of order 10^{-4} for the l^1 norm.

3   Existence of an attractor for the discrete model.

Proposition 3.1. *Let $\Lambda = \bigcap_{n \geq 0} T^n(\mathcal{K})$. Then $\Lambda \neq \emptyset$ is compact T-invariant and there is a neighborhood $U = U(\Lambda)$ such that $T(\overline{U}) \subset U$, i.e., Λ is an attractor for T.*

Remark 3.2. *We are not assuming that Λ is transitive in the definition of attractor.*

Proof. Since $T(\mathcal{K}) \subset \mathcal{K}$ we have that $C_n = \cap_{j=0}^n T^j(\mathcal{K})$ is a decreasing sequence of non empty compact subsets of $\mathbb{R}^{2p+1}$; $C_0 \supset C_1 \supset \cdots \supset C_n \supset \cdots$. Thus, by Baire Theorem, we have that $\Lambda \neq \emptyset$ and Λ is compact.

By definition of Λ we have

$$T(\Lambda) = T(\cap_{n\geq 0} T^n(\mathcal{K})) \subset \cap_{n\geq 0} T^{n+1}(\mathcal{K}) \subset \cap_{n\geq 0} T^n(\mathcal{K}) = \Lambda\,,$$

proving that Λ is T-invariant.

Let $[\mathcal{K}]_\epsilon := \{N \in \mathbb{R}^{2p+1} : \operatorname{dist}(N, \mathcal{K}) \leq \epsilon\}$ and $\epsilon > 0$ be so small that

$$[\mathcal{K}]_\epsilon \subset H = \{N = (N_0, N_1, \ldots, N_{2p}) \in \mathbb{R}^{2p+1} : \forall\, j = 0, 1, \ldots 2p : N_j > 0\}\,.$$

By Proposition 2.3 and Proposition 2.5, for all $x \in [\mathcal{K}]_\epsilon$ there is $n(x) > 0$ such that $T^{n(x)}(x) \in \mathcal{K}$. By continuity of T, see Lemma 2.6, there is $U(x)$ a neighborhood of x contained in H such that $T^{n(x)}(y) \in \mathcal{K}$ for all $y \in U(x)$. By compactness of $[\mathcal{K}]_\epsilon$ there is $n_1 > 0$ such that $T^n([\mathcal{K}]_\epsilon) \subset \mathcal{K}$ for all $n \geq n_1$.

Let now $U(\Lambda)$ be a neighborhood of Λ contained in $[\mathcal{K}]_\epsilon$.

Claim 3.1. *There is $n_0 > 0$ such that $T^n(\mathcal{K}) \subset U(\Lambda)$ for all $n \geq n_0$.*

Proof. The proof goes by contradiction. If it were not true, for all $j \in \mathbb{N}$ there would exist $x_j \in \mathcal{K}$ and $n_j > n_{j-1}$, such that $T^{n_j}(x_j) \notin U(\Lambda)$. Since $\mathcal{K}$ is compact there exists a convergent subsequence from $\{T^{n_j}(x_j)\}_{j\in\mathbb{N}}$. Without loss we may assume that $\{T^{n_j}(x_j)\}_{j\in\mathbb{N}}$ itself converges to a point $z \in \mathcal{K}$. Such a point z cannot be in Λ since $T^{n_j}(x_j) \notin U(\Lambda)$ for every $j \in \mathbb{N}$. But, since $T^{n+1}(\mathcal{K}) \subset T^n(\mathcal{K})$ for all $n \in \mathbb{N}$, we obtain $T^{n_j}(x_j) \in \cap_{h=0}^{n_j} T^h(\mathcal{K})$. Moreover, $z \in \cap_{h=0}^{n_j} T^h(\mathcal{K})$, otherwise there is $\epsilon > 0$ such that $\operatorname{dist}(z, \cap_{h=0}^{n_j} T^h(\mathcal{K})) > \epsilon$. But $\cap_{h=0}^{n_j} T^h(\mathcal{K}) \supset \cap_{h=0}^{n_{j+1}} T^h(\mathcal{K})$ for all $j \in \mathbb{N}$, so that $\operatorname{dist}(z, \cap_{h=0}^{n_{j+l}} T^h(\mathcal{K})) \geq \epsilon$ for every $l \geq 0$ contradicting the fact that $T^{n_{j+l}}(x_{j+l}) \to z$ when $l \to \infty$. It follows that $T^n(\mathcal{K}) \subset U(\Lambda)$ for all $n \geq n_0$, proving the claim. $\square$

To conclude the proof of the proposition it is enough to verify that there is $n_2 > 0$ such that $T^{n_2}(\overline{U(\Lambda)}) \subset U(\Lambda)$. This follows from the fact that $T^{n_0}(\mathcal{K}) \subset U(\Lambda) \subset \overline{U(\Lambda)} \subset [\mathcal{K}]_\epsilon$ taking $n_2 = n_0 + n_1$, thus Λ is an attractor. $\square$

It is clear that the fixed point p given by Corollary 2.7 belongs to Λ. In [Ar, Section B.8] by numerical methods it is found a candidate to be a fixed point. As we have pointed out above, in view of Corollary 2.7, the search for such a fixed point has sense.

Remark 3.3. *By Corollary 2.5, the basin of attraction of Λ is the whole set H of points with positive coordinates (see equation (6)). Moreover, since $\mathcal{K}$ is a disk, we can choose $U(\Lambda)$ simply connected in the proof of Proposition 3.1. These facts have some theoretical implications that we discuss in section 4.*

Lemma 3.4. *Assume that $S(2p - 1) > 0$ and $m_\rho(1) = 1$. Moreover also assume that T depends smoothly on $N = (N_0, N_1, \ldots, N_{2p})$ and that $\frac{\partial(N_j m(N_j))}{\partial N_j} \neq 0$, $j = 0, 1, \ldots, p$. Then the differential $D_N T : \mathbb{R}^{2p+1} \to \mathbb{R}^{2p+1}$ is a non singular linear map.*

Proof. Let us duplicate the $(p+1)$-th coordinate, N_p, of $N = (N_0, \ldots, N_p, \ldots, N_{2p})$, i.e., we write $\widehat{N} = (N_0, \ldots N_p, N_p, \ldots N_{2p}) = (N_{(0)}, N_{(1)})$, and consider $\widehat{T}(N_{(0)}, N_{(1)}) = (N_{(1)}, N_{(2)})$ where $N_{(2)} = (N_{2p}, \ldots, N_{3p})$. Thus, since the pth-coordinate equals the $(p+1)^{th}$-coordinate, $\widehat{T}(N_{(0)}, N_{(1)})$ is such that $\Pi_p(\widehat{T}(N_{(0)}, N_{(1)})) = T(N)$ and if $\widehat{T}$ is locally injective then T is locally injective too. Here $\Pi_p : \mathbb{R}^{2p+2} \to \mathbb{R}^{2p+1}$ is the projection

$$\Pi_p(x_0, \ldots, x_p, x_{p+1}, \ldots, x_{2p+1}) = (x_0, \ldots, x_{p-1}, x_{p+1}, \ldots, x_{2p+1})\,.$$

Taking into account that $(N_{2p}, \ldots, N_{3p})$ depends on $(N_0, \ldots, N_p, \ldots, N_{2p})$, this artifice allows us to write $\widehat{T}(N_{(0)}, N_{(1)}) = (N_{(1)}, F(N_{(0)}, N_{(1)}))$, and therefore

$$D\widehat{T} = \left(\begin{array}{ccc} A & | & Id \\ ---- & ---- & ---- \\ \frac{\partial F}{\partial N_{(0)}} & | & \frac{\partial F}{\partial N_{(1)}} \end{array} \right)$$

where A is a $(p+1) \times (p+1)$ matrix of the form

$$A = \left(\begin{array}{cccc} 0 & 0 & \cdots 0 & 1 \\ 0 & 0 & \cdots 0 & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots 0 & 0 \end{array} \right)$$

and Id is the identity $(p + 1) \times (p + 1)$ matrix.

To prove that $\widehat{T}$ is locally injective it suffice to prove that $\det D\widehat{T} \neq 0$. Hence, since $\det(A) = 0$ we are left to prove that $\det \left(\frac{\partial F}{\partial N_{(0)}} \right) \neq 0$. For this we proceed as follows. Using the expression for N_t given at equation (3) and denoting $\frac{\partial(N_j m(N_j))}{\partial N_j}$ by $h(N_j)$ we compute $\frac{\partial F}{\partial N_{(0)}}$ and find

$$\frac{1}{p} \begin{pmatrix} h(N_0)m_\rho(1)S(2p-1) & h(N_1)m_\rho(2)S(2p-2) & \dots\dots\dots & h(N_p)m_\rho(p)S(p) \\ 0 & h(N_1)m_\rho(1)S(2p-1) & \dots\dots\dots & h(N_p)m_\rho(p-1)S(p+1) \\ 0 & 0 & \dots\dots\dots & \dots\dots\dots \\ \dots\dots\dots & \dots\dots\dots & \dots\dots\dots & \dots\dots\dots \\ 0 & 0 & \dots\dots\dots & h(N_p)m_\rho(1)S(2p-1) \end{pmatrix} .$$

Since by hypothesis $h(N_j) \neq 0$ the thesis follows. $\qquad\square$

Corollary 3.5. *Under the hypothesis of Lemma 3.4 we have that $T : \Lambda \to \Lambda$ is locally injective.*

Remark 3.6. *Albeit $T : H \to H$ is locally injective, by Lemma 3.4, it is not globally injective. To see this assume that $N_{max} > 1$, $\gamma > 1$ and that the definition of $m(N)$ is given by equation (2). If $(N_0, N_1, \dots, N_{2p-1}) = (N_{max}, N_{max}, \dots, N_{max})$ then for $t = 2p, 2p+1, \dots, 2p + A_0p$ we get*

$$N_t = \frac{1}{p} \sum_{h=A_0 p}^{2p-1} N_{t-h} m(N_{t-h}) m_\rho(t-h) S(h) = \frac{1}{p} \sum_{h=A_0 p}^{2p-1} N_{max} m(N_{max}) m_\rho(t-h) S(h)$$

$$= \frac{1}{p} N_{max}^{1-\gamma} m_0 \sum_{h=A_0 p}^{2p-1} m_\rho(t-h) S(h) . \tag{9}$$

Similarly if we put $(N_0, N_1, \dots, N_{2p-1}) = (N_{max}^{1-\gamma}, N_{max}^{1-\gamma}, \dots, N_{max}^{1-\gamma})$ we obtain the same values for N_t. By induction we get that all values are the same for $t \geq 2p$ implying that T is not globally injective.

Let us point out that:

1. In the original model, [Ar], $m(N_j)$ is given by equation (2) i.e., $m(N_j) = m_0$ if $N_j \leq 1$ and $m(N_j) = m_0 N^{-\gamma}$ if $N_j > 1$. Hence $N_j m(N_j) = m_0 N_j$ if $N_j \leq 1$ and $N_j m(N_j) = m_0 N_j^{1-\gamma}$ if $N_j > 1$ implying that

$$h(N_j) = \frac{\partial(N_j m(N_j))}{\partial N_j} = \begin{cases} m_0 & \text{if} \quad N_j \leq 1 \\ m_0(1-\gamma)N_j^{-\gamma} & \text{if} \quad N_j > 1 \end{cases} .$$

   Since $\gamma > 1$, we have $h(N_j) \neq 0$ for all $N_j \neq 1$.

2. Assuming that T^2 is C^1, Lemma 3.4 gives that the fixed point p found at Corollary 2.7 has all its eigenvalues different from zero. The numerical approximation of the eigenvalues of DT_p^2, for the estimated value of p obtained by [Ar, Section 4.2.7] and our own estimates gives that there is a single eigenvalue of modulus greater than 1 which is negative, and there are $A_1 p$ eigenvalues of modulus less than 1. Hence p is a codimension one hyperbolic fixed point of T^2.

3. The hypothesis $S(2p - 1) \neq 0$ is reasonable: otherwise one can see that for two initial vectors $N = (N_0, N_1, \cdots, N_{2p})$ and $N' = (N_0', N_1, \cdots, N_{2p})$ with $N_0 \neq N_0'$ we get $T(N) = T(N')$. Thus the number $N_{A_1 p}$ of individuals at time $A_1 p$ is not affected by the first set N_0 of initial individuals. In another words the system looses memory for a number of years less than A_1 and so the actual dimension of the domain of T would be less than $A_1 p + 1$.

4   Study of Λ for $(A_0, \rho, \gamma) = (0.18, 0.30, 8.25)$.

In what follows we will assume that T is smooth (see [Ar, Section 2]) and that the calculations made for the parameter values $(0.18, 0.30, 8.25)$ are accurate enough to obtain that if p is the fixed point given by Corollary 2.7 then the eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_{2p}$ and μ of $D_p T$ satisfies $|\lambda_j| < 1$ for every $j = 1, \dots, 2p$, and $\mu \approx -3.335$, in particular $|\mu| > 1$[1]. Lemma 3.4 proves that p is in fact a hyperbolic fixed point with $W^s(p)$ being a codimension

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[1]Arlot in [Ar, Section 4.2.7], obtains that $\mu \approx -2, 29$ for the parameter values $(0.15, 0.30, 8.25)$.

one manifold and $W^u(p)$ an arc. Moreover, since Λ is an attractor, we have that $W^u(p) \subset \Lambda$ from which the fractal dimension of Λ is strictly greater or equal than 1. The calculations made in [Ar, Section 4.2.5] give for this fractal dimension a value around 1.33 from which Arlot conjectures that locally the attractor is the product of a line by a Cantor set.

Here we shall discuss if for the choice $A_0 = 0.18$, $\rho = 0.30$ and $\gamma = 8.25$ the system given by T can be transitive.[2]

Definition 4.1. *Let $f : X \to X$ be a continuous map defined in the topological space X. We say that the system defined by f is (topologically) transitive if for every pair of non-empty open subsets A, B of X there is $n \in \mathbb{Z}$ such that $f^n(A) \cap B \neq \emptyset$. The dynamical system defined by f is topologically mixing if for every pair of non-empty open subsets A, B of X there is $N > 0$ such that $f^n(A) \cap B \neq \emptyset$ for all $n \geq N$.*

In [Ar, Section 5] it is pointed out the interest in studying the case where the parameters are $A_0 = 0.18$, $\rho = 0.30$, $\gamma = 8.25$: it is because the numerical simulations indicates that for this parameter choice $T_{|\Lambda}$ is transitive, see [Ar, Section 4.1.3, figure 12]. Moreover, in [Ar, Section 4.2.7, figures 34 and 35] the geometry of the attractor Λ is depicted from the successive iterates of the local unstable manifold of the fixed point p. This suggests that $W^u(p)$ is dense in Λ. This was confirmed by the numerical simulations done by us, see figure 2. The next proposition shows that if the orbit of a point in $W^u(p)$ is dense in Λ then $T_{|\Lambda}$ is in fact topologically mixing.

Proposition 4.1. *Let us assume that there exists $x_0 \in W^u(p)$ such that $clos(orbit^+(x_0)) = \Lambda$ that there exists a homoclinic point x for p that we do do not have tangencies between the stable and unstable manifold of p and that forward iterates by T^2 of an unstable segment $s \subset W^u(p)$ has diameter bounded away from zero. Then $T : \Lambda \to \Lambda$ is topologically mixing.*

Proof. Observe that by hypothesis we have in particular that $clos W^u(p) = \Lambda$. Let $A \neq \emptyset$ and $B \neq \emptyset$ be open subsets of Λ, i.e., there are open subsets $\mathcal{A}$ and $\mathcal{B}$ of $\mathbb{R}^{2p+1}$ such that $A = \mathcal{A} \cap \Lambda$ and $B = \mathcal{B} \cap \Lambda$. We will prove that there exists n_0 such that for all $n \geq n_0$ we have $T^n(A) \cap B \neq \emptyset$ thus proving that T is topologically mixing. Since $W^u(p)$ is dense in Λ there is $n_2 > 0$ such that $T^{n_2}(x_0) \in \mathcal{A}$. Thus $W^u(p)$ cuts $\mathcal{A}$ in an arc s containing $T^{n_2}(x_0)$. Since $orbit(x_0)$ is dense in Λ there exists $n_1 > n_2$ such that $T^{n_1-n_2}(x_0) \in U(p)$ where $U(p)$ is a neighborhood of p in which we may assume that we have C^1-linearizing coordinates, and $T^{n_1-n_2}(s)$ contains an arc J which intersects transversally $W^s_{loc}(p)$, this follows from the assumptions we have done. By the Inclination Lemma, see [PM, Chapter 2, §7], $T^n(J)$ C^1-approaches on compact segments of $W^u(p)$. Let $\nu > 0$ be the radius of a ball contained in $\mathcal{B}$. There is $n_0 > n_1$ such that $T^{n_0}(J)$ is $\nu/2$-dense in Λ and hence $T^n(J)$ is $\nu/2$-dense in Λ for all $n > n_0$. Thus $T^n(J)$ cuts $\mathcal{B}$ implying that $T^n(\mathcal{A}) \cap \mathcal{B} \neq \emptyset$ for $n \geq n_0$. But since $W^u(p) \subset \Lambda$ (Λ is an attractor) we conclude that $T^n(A) \cap B \neq \emptyset$ for $n \geq n_0$ proving that T is topologically mixing. $\square$

Remark 4.2. *Roughly speaking the above result means that for the parameter values $A_0 = 0.18$, $\rho = 0.30$ and $\gamma = 8.25$, from the topological viewpoint we have that all possible states $(N_0, N_1 \ldots, N_{2p}) \in \Lambda$ are visited and so a chaotic behavior should be expected. On the other hand, since there are fixed points like p in Λ if $(N_0, N_1 \ldots, N_{2p})$ is very near p in practice we will see the same behavior for large periods of time seeming that the population of these rodents is in equilibria. On the other hand the hypothesis we have assumed seems to be rather strong.*

Another consequence of the density of the unstable manifold of p in Λ is the following (see also Remark 6.1).

Proposition 4.3. *If $clos(W^u(p)) = \Lambda$ then $T^2_{|\Lambda} : \Lambda \to \Lambda$ is injective.*

Proof. Indeed, T^2 is injective when restricted to $W^u(p)$, for, if it were not true, there would exist $x, y \in W^u(p)$ such that $T^2(x) = T^2(y)$. But, since $T^2(p) = p$ it holds that $W^u(p) = \cup_{n \in \mathbb{N}} T^2(W^u_\varepsilon(p))$ where $W^u_\varepsilon(p)$ is the ε-local-unstable manifold of p. Thus there is $N > 0$ such that $x, y \in T^{2N}(W^u_\varepsilon(p))$ and, hence, there is an arc $\gamma \subset W^u(p)$ with end points x and y. Applying T^2 to γ we find a closed loop $T^2(\gamma)$ contained in $W^u(p)$ which contradicts the fact that $W^u(p)$ is homeomorphic to $\mathbb{R}$.

Assume now that there are $x, y \in \Lambda$ such that $T^2(x) = T^2(y)$. Since T^2 is locally injective there is $r_1 > 0$ such that $y \notin B(x, r_1)$ where $T^2_{|B(x,r_1)} : B(x, r_1) \to H$ is a homeomorphism. There exists also $r_2 > 0$ such that $T^2_{|B(y,r_2)} : B(y, r_2) \to H$ is a homeomorphism. Hence we may find $V(x) \subset B(x, r_1)$ a neighborhood of x and $V(y) \subset B(y, r_2)$ a neighborhood of y such that $T^2(V(x)) = T^2(V(y))$. Since, by assumption, $W^u(p)$ is dense in Λ, there is an arc $\gamma \subset W^u(p)$ such that has its end points $x' \in V(x)$ and $y' \in V(y)$ such that $T^2(x') = T^2(y')$ contradicting that $W^u(p)$ is homeomorphic to $\mathbb{R}$. $\square$

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[2] We thank Enrique Pujals for fruitful discussions on this topic.

We point out that the numerical simulations presented in the appendices justify that the hypothesis assumed in Propositions 4.1 and 4.3 are reasonable. Indeed we found:

1. If there is a homoclinic point we must have positive entropy. We estimate in Appendix A the order-2 Kolmogorov entropy of the attractor, [Ta], and found a positive value ≈ 0.75.

2. The absence of tangencies should be checked in a certain way, at least in a neighborhood of p. In algorithm "homclin4" presented in Appendix B, we compute the angle between the local unstable manifold $W^u_\epsilon(p)$ and the iterate $T^m(\ell)$, of an arc $\ell \subset W^u_\epsilon(p)$, for $m > 0$ such that $T^m(\ell)$ is near p, founding in all cases values close to π or 0 radians, thus $W^u_\epsilon(p)$ and $T^m(\ell)$ are almost parallel.

3. That there is a point in $W^u(p)$ whose orbit is dense is a rather strong assumption. But when we plot the image of the first 1000 iterates of a single point of the local unstable manifold $W^u_{loc}(p)$, projected into $I\!R^3$ we roughly recover the image of Λ obtained plotting all the sequences of points pseudo-randomly generated, see Appendix F. Moreover, in all the simulations done in algorithm "entropia3" presented in Appendix B, we always obtain that if $N \neq N'$ then $T^2(N) \neq T^2(N')$, indicating that the hypothesis of the density of $W^u(p)$ in Λ assumed in Propositions 4.1 and 4.3 is consistent.

4. That forward iterates of a non trivial segment $s \subset W^u(p)$ have their diameters bounded away from zero also is rather strong. But again in all the simulations done, in particular in all runs of algorithm "homclin4", presented in Appendix B, we verify that this is the case.

5. Moreover, there are theoretical results that point out that in a setting like that of this model, we cannot expect $T^2_{|\Lambda}$ to be C^1-robustly transitive. Indeed, by construction the attractor Λ is contained in a simply connected neighborhood $U \subset I\!R^{201}$. Then by a C^1-small perturbation we may create a sink (see [RS] for instance) whose basin of attraction may contain (part of) $W^u(p)$. Nevertheless, the type of perturbations we can perform with T is not arbitrary and so we cannot reject *a priori* that for certain parameter values (like $(A_0, \rho, \gamma) = (0.18, 0.30, 8.25)$) the system is transitive.

In the following subsections we check numerically the hypothesis of Propositions 4.1 and 4.3.

4.1 Estimation of the Kolmogorov Entropy of the Attractor

As a first step to estimate the presence of chaos in Λ is to verify that it has sensibility with respect to initial data. To do so we have made computer simulations of the system given by (3) with the parameter values $(0.18, 0.30, 8.25)$. That Λ presents sensibility to initial conditions has been pointed out by Arlot, [Ar, Section 4.2.6]. To test this property we proceed as follows:

1. We generate M independent initial vectors $N^{(j)} = (N_0^{(j)}, N_1^{(j)}, \ldots, N_{A_1 p}^{(j)})$, $1 \leq j \leq M$. In fact what we have done is to generate $M = 400$ files with initial data chosen in a pseudo-random way. We assume that these 400 initial data are independent.

2. We iterate ℓ-times by T^2 so that $T^{2\ell}(N^{(j)})$ can be assumed, from the practical point of view, to belong to the attractor. The value of ℓ that we have chosen is $\ell = 10000$ so that we are considering $T^{20000}(N^{(j)})$. For simplicity of notation we still denote this iterate by $N^{(j)}$.

3. We add a small noise $\Delta N^{(j)}$ to $N^{(j)}$ obtaining $\widetilde{N}^{(j)} = N^{(j)} + \Delta N^{(j)}$. In the computer simulations we choose $10^{-10} \leq \|\Delta N^{(j)}\| \leq 10^{-8}$.

4. We specify a initial distance d_0 and compute for every j the integer b_j such that

$$\|T^{2i}(N^{(j)} - T^{2i}(\widetilde{N}^{(j)})\| \leq d_0, \, 0 \leq i < b_j, \quad \text{and} \quad \|T^{2b_j}(N^{(j)} - T^{2b_j}(\widetilde{N}^{(j)})\| > d_0 \,.$$

We choose $d_0 = 0.1$ since we observe fast divergence between the orbits when this distance is achieved.

5. In all the simulations we have done we find that $b_j \leq 80$. In fact, we change the size of the perturbation finding that even with $10^{-18} < \|\Delta N^{(j)}\| \leq 10^{-16}$, the value of b_j satisfies $b_j \leq 200$. We conclude that there are numerical evidences that $T_{|\Lambda}$ exhibits high sensibility to initial conditions.

As a second step to test the chaotic behavior on Λ we estimate its order-2 Kolmogorov entropy K giving by the average time for two initially near orbits of the attractor to diverge. More precisely, K is calculated from the average time t_0 that is needed for two points in the attractor, which are initially within a specified maximum distance d_0, to separate until the distance between these points has become larger than d_0.

The Kolmogorov entropy of an attractor can be considered as a measure for the rate of information loss along the attractor or as a measure for the degree of predictability of points along the attractor given an initial data.

In general, a positive entropy is considered as the conclusive proof that the dynamical system is chaotic. A zero entropy represents a constant or a regular phenomena that can be represented by a fixed point or a periodic attractor, [Ta].

Here we apply the definitions of the order-2 Kolmogorov entropy suggested by Takens in [Ta] and by Grassberger and Procaccia in [GP], see also [GP2]. According to these definitions, we will estimate the entropy from the average time required for two nearby distinct orbits of the attractor to diverge.

According to Takens [Ta] and Grassberger and Procaccia [GP], the separation of distinct nearby orbits is assumed to be exponential and the time interval t_0 required for two initially nearby points to separate by a distance larger than d_0 will be exponentially distributed according to

$$C(t_0) \sim e^{-Kt_0},$$

where K is the Kolmogorov entropy, see [GP3]. For practical purposes $C(t_0)$ may be transformed into a discrete distribution function defined as

$$C(b) = e^{-Kb\tau_s}, \quad \text{with} \quad b = 1, 2, 3, \dots,$$

where τ_s is the time step between two sampled data points. Given an initial pair of independent points within a distance d_0, the variable b is the number of sequential pairs of points on the attractor such that the interpoint distance is for the first time bigger than d_0.

To estimate K we proceed as follows.

1. We generate Z independent initial vectors $N^{(j)} = (N_0^{(j)}, N_1^{(j)}, \dots, N_{A_1 p}^{(j)})$, $1 \leq j \leq Z$. For practical purposes we take for Z the same $M = 400$ files used to estimate sensibility to initial conditions.

2. We iterate ℓ-times by T^2 so that $T^{2\ell}(N^{(j)})$ can be assumed, from the practical point of view, to belong to the attractor. The value of ℓ that we have chosen is $\ell = 10000$ so that we are considering $T^{20000}(N^{(j)}) \in \Lambda$. For simplicity we still denote this iterate by $N^{(j)}$ and will denote the initial $N^{(j)}$ by $T^{-20000}(N^{(j)})$, but this is just a notation; we are not claiming that T is globally invertible.

3. For each $j = 1, \dots, Z$, we write in the file number j the values of

$$T^{-20000}(N^{(j)}), \ N^{(j)}, \ T^2(N^{(j)}), \ T^4(N^{(j)}), \dots, T^{2044}(N^{(j)}).$$

4. Given a distance $d > 0$, we search for pairs of vectors $T^{h_j}(N^{(j)})$, $T^{h_i}(N^{(i)})$ such that $\|T^{h_j}(N^{(j)}) - T^{h_i}(N^{(i)})\| < d$. According to [STB] the value of d should be smaller than $\frac{1}{100}$ of the absolute deviation δ_N The simulations we have done give that the mean value $< N >$ of the population N_t is about 2.335 and the average absolute deviation

$$\delta_N = \frac{1}{400 \times 2046 \times 200} \sum_{h,j,i} \left| T^h(N^j)_i - < N > \right| \approx 0.97,$$

   thus, we take $d \leq 0.97 \times 10^{-2}$ (the greater value of d we have used is $d = \frac{1}{128}$).

5. Given d, $T^{h_j}(N^{(j)})$ and $T^{h_i}(N^{(i)})$ as in item 4. above, we compute the integer $b = b(i, j, h_i, h_j)$ such that

$$\|T^{2s}(T^{h_j}(N^{(j)})) - T^{2s}(T^{h_i}(N^{(i)}))\| \leq d, \quad 0 \leq s < b, \quad \text{and}$$

$$\|T^{2b}(T^{h_j}(N^{(j)})) - T^{2b}(T^{h_i}(N^{(i)}))\| > d.$$

6. Letting $M = M(d)$ be equal to the number of distinct pairs

$$T^{h_j}(N^{(j)}), \ T^{h_i}(N^{(i)}), \ 1 \leq j < i \leq 400,$$

   verifying item 4. we compute $\bar{b} = \frac{1}{M} \sum_{j=1}^{M} b_j$. The program doing this task has to take care to not duplicate the number of times a given pair $T^{h_j}(N^{(j)})$, $T^{h_i}(N^{(i)})$ is computed and also to not consider as different strings the one starting at $s = 0$

$$\|T^{2s}(T^{h_j}(N^{(j)})) - T^{2s}(T^{h_i}(N^{(i)}))\| \leq d, \ 0 \leq s < b, \quad \text{and}$$

$$\|T^{2b}(T^{h_j}(N^{(j)})) - T^{2b}(T^{h_i}(N^{(i)}))\| > d,$$

   with the sub-strings starting at $s = s_0 > 0$

$$\|T^{2s}(T^{h_j}(N^{(j)})) - T^{2s}(T^{h_i}(N^{(i)}))\| \leq d, \ 0 < s_0 \leq s < b, \quad \text{and}$$

$$\|T^{2b}(T^{h_j}(N^{(j)})) - T^{2b}(T^{h_i}(N^{(i)}))\| > d.$$

7. Finally we estimate the value of the entropy K of T^2 by

$$\widehat{K} = -\frac{1}{\tau_s} \ln \left| 1 - \frac{1}{b} \right| ,$$

where $\widehat{K}$ is the maximum-likelihood estimate of the entropy K (see [STB]).

8. We repeat the items above for several values of d. Taking $d \approx 1/100$ we find more than 2000 verifying item 4., while for values of $d < 1/50000$ the number of such pairs is too low, less than 100. More precisely, for $d = 1/65536 = 0.0000152587890625$ we find 53 strings. This is reflected in the estimate of the standard deviation of the entropy: for values of d too small the sample is also small and the estimation of K is less accurate, as one can see in Appendix A.

To test a confidence interval for the values obtained to $\widehat{K}$ we need to estimate its standard deviation. For this note that the standard deviation of $\widehat{K}$ can be obtained from the variance of b. To do so recall, [STB], that

$$var(b) = \frac{e^k}{(e^k - 1)^2}, \quad \text{where} \quad k = K\,\tau_s .$$

The standard deviation in the estimate of $\bar{b}$, computed in item 6. is given by

$$\sigma(\bar{b}) = \sqrt{var(b)/M} = \frac{e^{\widehat{k}/2}}{\sqrt{M}(e^{\widehat{k}} - 1)} .$$

For large values of M, $\sigma(\bar{b})$ will be small. In that case we can use the derivative of the function $k = -\ln(1 - 1/b)$ in the point $\widehat{k} = \widehat{K}\tau_s$ to estimate the standard deviation of k.

The values obtained for the entropy of T^2 are listed in two tables in Appendix A which contain also the values of d we have used and those of the standard deviation σ_K of the entropy. For both extreme values used for d, namely $d = 1/128 = 0.0078125$ and $d = 1/65536 = 0.0000152587890625$ the results are less accurate, since 0.0078125 is "too big" with respect to $0.97 \approx 1$, and for 0.0000152587890625 there are few sample points, see [ER].

Nevertheless all the estimates obtained show that $T^2_{|\Lambda}$ has positive entropy, which implies that $T_{|\Lambda}$ also has positive order-two entropy $K \approx 0.37$.

Thus we have strong numerical evidence that Λ is a chaotic attractor.

Remark 4.4. *We do not claim that we have estimated the entropy of $T_{|\Lambda}$. The calculations made has to be seen as an indication that the model given by equation (3) exhibits a chaotic behavior. Rigorous proofs are needed to confirm our estimations.*

5 Existence of homoclinic points: numerical approach.

In dynamical systems the presence of chaotic behavior is often associated to the existence of homoclinic points. We have assumed their existence in Proposition 4.1 to obtain that Λ is topologically mixing. Next we check numerically their existence. To do it we proceed as follows:

5.1 Approximated $W^u_{loc}(p)$.

Due to the fact that $W^u(p)$ is one dimensional a first attempt is to try to pick a fundamental domain in $W^u_{loc}(p)$ and search by brute force if it is possible to find a candidate to be a homoclinic point there. Problem: we do not know precisely $W^u_{loc}(p)$. Moreover, the value of the fixed point p is known only by an approximate value $\hat{p}$. But we know that there are only one eigenvalue μ of modulus greater than 1 of DT^2_p and μ is negative. Hence, since the other A_1p eigenvalues are small in modulus, in fact all of them have modulus less than 0.5, we may assume that $W^s_{loc}(p)$ is a A_1p-dimensional disk so if we iterate $\hat{p}$ by T^2, since $\mu < 0$ we have that the segment $[\hat{p}, T^2(\hat{p})]$ cuts $W^s_{loc}(p)$ at a unique point. By the λ-lemma we have that the successive iterates of $[\hat{p}, T^2(\hat{p})]$ by T^2 converges to $W^u_{loc}(p)$.

Thus for numerical simulations we can take as $W^u_{loc}(p)$ one of these segments. In some of our simulations we choose $[T^{38}(\hat{p}), T^{40}(\hat{p})]$ as $W^u_{loc}(p)$ and in others we take $W^u_{loc}(p)$ as $[T^{30}(\hat{p}), T^{32}(\hat{p})]$. Observe that the length of $[T^{38}(\hat{p}), T^{40}(\hat{p})]$ is less than 10^{-3} and the length of $[T^{30}(\hat{p}), T^{32}(\hat{p})]$ is less than 10^{-4}. Hence, since the mean value of the data is 2.335 and that the absolute deviation is 0.97 such lengths are relatively small.

We subdivide the chosen segments in 10000 equal parts and iterate more than 2000 times by T every point y of the subdivision finding the iterate $T^{2j}(y)$ closer to $\hat{p}$. In order to not consider misleading solutions, we discard the first 20 iterates and check that the orbit of y is "returning near the point $\hat{p}$", i.e., we check that the minimum distance is not achieved in the 21th iterate. Then we create a table containing the values of y and of the iterate of y closer to $\hat{p}$. Not that this procedure does not prove that any of such a point y is a homoclinic point.

5.2   Returning points.

After this we find the value of y_0 and j_0 that minimizes $\text{dist}(T^{2j}(y), \hat{p})$. In the simulations corresponding to $W_{loc}^u(p) \approx [T^{38}(\hat{p}), T^{40}(\hat{p})]$ we find that

$$y_0 = T^{38}(\hat{p}) + \frac{5102}{10000}\left(T^{40}(\hat{p}) - T^{38}(\hat{p})\right) \quad \text{and} \quad j_0 = 629\,.$$

We find a suitable sub-interval I_0 such that $y_0 \in I_0 \subset I \subset [T^{38}(\hat{p}), T^{40}(\hat{p})]$, we iterate 10 times by T^2 the point y_0 and the extreme points of the segment I_0, calling them L_0 and R_0 [3]. After this we subdivide again $T^{20}(I_0)$ and find a small interval $I_1 \subset T^{20}(I_0)$ around $T^{20}(y_0)$ and iterate again their end points L_1, R_1 and also $T^{20}(y_0)$. We continue with this procedure finding segments $I_h \subset T^{20}(I_{h-1})$ and their end-points L_h, R_h till we arrive to the value of j_0. There are cases that we cannot iterate 10 times by T^2 because distances become relatively large or because we cannot assume $T^{20}([L_h, R_h])$ to be a straight segment and in that cases we reduce the step size. The final step does not have to be a multiple of 10. We found that a suitable value for the length of the initial segment I_0 is 1.122×10^{-7}. To validate this procedure we have to check several things:

1. control that the length of $T^{20}(I_h)$ does not increase too much: we do not accept a length greater than 10^{-4}. If the length of $T^{20}(I_h)$ is greater than 10^{-4} we reduce the step used: first to 8 iterates by T^2 and finally by 2 iterates by T^2. In our computations we do not need to further reduce this number of iterates.

2. control that the segment $T^{20}(I_h)$ (or $T^{16}(I_h)$ or $T^4(I_h)$ in case that we have to choose a smaller step) does not bend too much: we require that $T^{20}(I_h)$ behaves like a straight segment. To do so we subdivide the segment I_h into four equal smaller segments $[L_h, L_h'], [L_h', T^{20h}(y_0)], [T^{20h}(y_0), R_h'],$ and $[R_h', R_h]$. Next we check that after 10 iterates of these intervals by T^2, the sum of their lengths satisfies that

$$T^{20}([L_h, L_h']) + T^{20}([L_h', T^{20h}(y_0)]) + T^{20}([T^{20h}(y_0), R_h']) + T^{20}([R_h', R_h])$$

is almost the same as the length of $T^{20}([L_h, R_h])$. We reject any case where the quotient between both quantities is greater than 1.0001, reducing the number of iterates if it were necessary[4].

5.3   Far from tangencies.

After computing $T^{j_0}(y_0)$ and the corresponding points L_{h_0} and R_{h_0} for suitable h_0 [5] we compute the angle between $[L_{h_0}, R_{h_0}]$ and $[T^{38}(\hat{p}), T^{40}(\hat{p})]$. We expect to have an angle close to 0 or 180 degrees, and in fact this is the case in all the simulations: we obtain for the angle the value of 3.108×10^{-5} radians. This is an indication that we are not near a tangency.

5.4   Evidence of homoclinic points.

1. For a suitable choice of $I_0 = [L_0, R_0]$ we compute the angle between the segments $[\hat{p}, L_{h_0}]$ and $[\hat{p}, R_{h_0}]$. This is a key point in our calculations. Before we indicate how we proceed to do so, recall that the codimension one submanifold $W_{loc}^s(p)$ of $\mathbb{R}^{A_1p+1}$ locally separates $\mathbb{R}^{A_1p+1}$ in two regions that we denote by $W^{s,+}$ and $W^{s,-}$.

   On the one hand, if $L_{h_0} \in W^{s,+}$ and $R_{h_0} \in W^{s,-}$ then $[L_{h_0}, R_{h_0}]$ intersects $W_{loc}^s(p)$ and so we have a homoclinic point in this segment $[L_{h_0}, R_{h_0}]$. Hence, by the λ-lemma the angle between successive iterates of the vectors $[\hat{p}, L_{h_0}]$ and $[\hat{p}, R_{h_0}]$ would increase up to a value close to π.

---

[3]To try to subdivide the interval $I \subset [T^{38}(\hat{p}), T^{40}(\hat{p})]$ around y_0 of end points $T^{38}(\hat{p}) + (5101/10000)(T^{40}(\hat{p}) - T^{38}(\hat{p}))$ and $T^{38}(\hat{p}) + (5103/10000)(T^{40}(\hat{p}) - T^{38}(\hat{p}))$ to obtain more precision is not a good idea since forward iterates by T^2 of I increases their length exponentially fast. We loose any precision in the calculus after less than 20 iterations by T^2.

[4]In fact at the scale we have chosen this has never been the case for reasonable values of $\ell([L_h, R_h])$.

[5]If the number of iterates is always 10 then we get $h_0 = \left[\frac{j_0}{10}\right]$.

On the other hand, if both points are in the same region with respect to $W_{loc}^s(p)$, say $L_{h_0}, R_{h_0} \in W^{s,+}$, then the segment $[L_{h_0}, R_{h_0}]$ will not cut $W_{loc}^s(p)$ and, again by the λ-lemma, we have that the angle between successive iterates of the vectors $[\hat{p}, L_{h_0}]$ and $[\hat{p}, R_{h_0}]$ goes to zero when we iterate by T^2. In this case the existence of a homoclinic point cannot be guaranteed.

In the simulations we have done, see Appendix F, we obtain that for I_0 of length 1.122×10^{-7} the initial angle between $[\hat{p}, L_{h_0}]$ and $[\hat{p}, R_{h_0}]$ is 1.337 radians, approximately 77 degrees. For the angle between $[\hat{p}, T^2(L_{h_0})]$ and $[\hat{p}, T^2(R_{h_0})]$ we obtain a value of 3.011 radians which is about 173 degrees. For the angle between $[\hat{p}, T^4(L_{h_0})]$ and $[\hat{p}, T^4(R_{h_0})]$ we obtain a value of 3.139 radians which is about 180 degrees and for the angle between $[\hat{p}, T^6(L_{h_0})]$ and $[\hat{p}, T^6(R_{h_0})]$ we obtain a value of 3.140 radians. For the subsequent iterates the angle diminishes slightly but up to the 14th iterate we find that the angle is close to π. Thus in that case we find evidence that a homoclinic point exists.

2. There are choices for the length of I_0 that does not lead to such evidence. Due to the exponential dilation in the unstable direction the behavior is rather sensible to this value. If we choose $\ell(I_0) = 1.046 \times 10^{-7}$, instead of 1.122×10^{-7}, we obtain at the final step that for this value both L_{h_0} and R_{h_0} belong to the same local connected component of $I\!R^{A_1 p+1} \backslash W_{loc}^s(p)$. In this case we have that the angle between $[\hat{p}, L_{h_0}]$ and $[\hat{p}, R_{h_0}]$ is 1.358×10^{-3} radians, the angle between $[\hat{p}, T^2(L_{h_0})]$ and $[\hat{p}, T^2(R_{h_0})]$ is 2.922×10^{-5} and the angle between $[\hat{p}, T^4(L_{h_0})]$ and $[\hat{p}, T^4(R_{h_0})]$ is 4.582×10^{-6}. This indicates that both points belong to the same region with respect to $W_{loc}^s(p)$. Thus we cannot ensure the existence of homoclinic points in this case.

But as we have shown above, there are choices for the length of I_0, subject to all the mentioned restrictions, that render numerical evidence that we in fact do have a homoclinic point associated to the fixed point p.

In the Appendix D we give the pseudo-code of the algorithms employed to test the existence of homoclinic points.

In Appendix F we show the values of the approximate homoclinic point $y_0 \in W_{loc}^u(p)$ and the angular values for the iterates $[\hat{p}, T^{2j}(L_{h_0})]$ and $[\hat{p}, T^{2j}(R_{h_0})]$ for $j = 0, 1 \ldots, 7$.

Acknowledgements

6 Appendices.

6.1 Appendix A: numerical results for the entropy.

The following tables gives the estimation of K_{T^2} with d varying from $d = 1/128$ to $d = 1/2048$ and d varying from $d = 1/4096$ to $d = 1/65536$ respectively. The values of d are evenly distributed.

entropy estimated	standard deviation	d of the estimation
0.718689733929300086	0.019083966799452565	0.0078125
0.72369632526302973	0.01900037121106015	0.00732421875
0.72434809650524738	0.019137203419564205	0.0068359375
0.71294980785612502	0.019363360739997387	0.00634765625
0.72339347917167310	0.019254622891041776	0.005859375
0.72850976773398998	0.019582964739043648	0.00537109375
0.72335976581552290	0.019863263097474933	0.0048828125
0.72087894452782916	0.020143892470160681	0.00439453125
0.73055620248396048	0.020338205554372255	0.00390625
0.73804687146973547	0.020719763256481047	0.00341796875
0.75191849797535868	0.021276865360116811	0.0029296875
0.73746684855086799	0.022038110478093715	0.00244140625
0.74278320582790917	0.022619630761394613	0.01953125
0.74484051347505878	0.023688490322879650	0.00146484375
0.75258445595582577	0.025936858832676262	0.0009765625
0.77155883257911018	0.030932730422267861	0.00048828125

entropy estimated	standard deviation	d of the estimation
0.79719104033302477	0.038944604944862542	0.000244140625
0.80273603232729273	0.040161477427822390	0.0002288818359375
0.80026054504518599	0.040945123242474202	0.000213623046875
0.80048080034198759	0.042277355365906191	0.0001983642578125
0.78529717602382290	0.043744279577575359	0.00018310546875
0.78083504948163185	0.045414270447522060	0.0001678466796875
0.79617940645818047	0.047304274588613750	0.000152587890625
0.80929969245707451	0.049841776725230737	0.0001373291015625
0.80683413899432532	0.053337136165239741	0.0001220703125
0.83598552255847603	0.056494298338862138	0.0001068115234375
0.80841322520717467	0.061957796245919238	0.000091552734375
0.86342039883772544	0.068153726594861803	0.0000762939453125
0.85991632434641512	0.076013458407264910	0.00006103515625
0.90006789636726776	0.091761409503824907	0.0000457763671875
0.79158725337319783	0.122667989144921260	0.000030517578125
0.96758402626170560	0.186693997202307350	0.0000152587890625

6.2 Appendix B: description of algorithms.

Taking into account [TR], we do not care so much about the embedding dimension and use directly as vectors of data those given by $N = (N_0, N_1, \ldots, N_{200})$.

- A first algorithm called "ratones" is used to generate 400 files named datos[i] $i = 1, 2, \ldots, 400$, each of which contains the following data:

  1. A random seed is generated to initialize a pseudo-random generator.

  2. For each i from 1 to 400 an initial vector of dimension 201 in which every component is a real number N_h. This real number N_h is in fact a floating point number of 80 bits following IEEE 754-1985[6] standards for the representation, calculations and manipulations of real numbers in a computer. The value of every element N_h for $h = 0$ to $h = 199$ is generated calling the RANDOM function available in the Software Library. The value of N_{200} is calculated from equation (3). $N^{init} = (N_0, N_1, \ldots, N_{200})$ is stored as the first value in the corresponding file datos[i].

  3. From equation (3) we compute the different values of N_h for $h \geq 201$, defining in this way recursively

$$T^2(N^{init}),\ T^4(N^{init}),\ T^6(N^{init}),\ \ldots.$$

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[6] IEEE Standard for Binary Floating-Point Arithmetic (ANSI/IEEE Std 754-1985). Also known as IEC 60559:1989, Binary floating-point arithmetic for microprocessor systems.

We discard the first 9999 iterates and stored in datos[i] the following 1024 ones,

$$T^{20000}(N^{init}).\, T^{20002}(N^{init}), \ldots, T^{22046}(N^{init}).$$

- A second algorithm that we call "ratones1" is used to perturb randomly $T^{20000}(N^{init})$ in each of the 400 files generated by "ratones" obtaining a vector $\widetilde{N}$. The random perturbations done vary from $-2^{-50} \approx -10^{-15}$ to $2^{-50} \approx 10^{-15}$ in each of the h-coordinates of $T^{20000}(N^{init})$ for h from 0 to 199. $\widetilde{N}_{200}$ is computed from equation (3).

- The third algorithm we use, called "sensible", computes for each i from 1 to 400 the number b_i such that for $j = 0$ to $j = b_i - 1$

$$\|T^j(T^{20000}(N^{init})) - T^j(\widetilde{N})\| \le 0.1 \quad \text{and} \quad \|T^{b_i}(T^{20000}(N^{init})) - T^{b_i}(\widetilde{N})\| > 0.1$$

We use the supremum norm in the calculations since this accelerate the computations and it is clear that the results do not depend on the norm used.

- Algorithm, "sensible", also computes the mean value $< b >$ of b_i as

$$< b > = \frac{1}{400} \sum_{i=1}^{400} b_i \, ,$$

in all the simulations done the value of b_i was less than 180 and $< b > \approx 100$.

- The forth algorithm, "dispersion", calculates the mean value $< N^{(i)} >$ of data stored in the files datos[i]. It calculates also the mean value of all data which gives a result of $< N > \approx 2.34$.

- Algorithm "dispersion" also computes the absolute average deviation

$$\delta_N = \frac{1}{400 \times 2046 \times 200} \sum_{h,j,i} \left| T^h(N^j)_i - < N > \right| \approx 0.97 \, .$$

- Given a value $d > 0$ the algorithm "entropia3" compares the data stored in datos[j] with that stored in datos[i] discarding the initial vectors (only after 20000 iterates by T we assume that the vectors are in Λ). For $1 \le j < i \le 400$ "entropia3" searches for pairs $T^{h_j}(N^{(j)})$, $T^{h_i}(N^{(i)})$ such that their distance, given by the norm of the supremum, is less d. "entropia3" runs 32 times generating 32 files named info[k], $k = 1, \ldots, 32$, of records each of which contains

  1. The number i of file datos[i],
  2. the number of iterates h_i by T from $N^{(i)}$,
  3. the value of $T^{h_i}(N^{(i)})$,
  4. the number j of file datos[j],
  5. the number of iterates h_j by T from $N^{(j)}$,
  6. the value of $T^{h_j}(N^{(j)})$.

For values of d not so small we obtain huge files info[k], and as d decreases the size of these files decreases. For computational reasons we choose $d_{max} = 1/128$ (corresponding to info[1] with $6{,}602\,KB$) and $d_{min} = 1/65536$ (corresponding to info[32] with $196\,KB$). Of course the files info[k] contain a lot of redundant information since if dist($T^{h_j}(N^{(j)})$, $T^{h_i}(N^{(i)})$) $< d$ and also dist($T^{h_j+l}(N^{(j)})$, $T^{h_i+l}(N^{(i)})$) $< d$, with $l > 0$ less than the least positive value b such that dist($T^{h_j+b}(N^{(j)}), T^{h_i+b}(N^{(i)})$) $\ge d$, we are storing $(j, h_j, T^{h_j}(N^{(j)}); i, h_i, T^{h_i}(N^{(i)}))$, and also $(j, h_j + l, T^{h_j+l}(N^{(j)}); i, h_i + l, T^{h_i+l}(N^{(i)}))$.

- Finally the algorithm "entropia4" computes the estimation of the second order entropy, $\widehat{K}$, and its standard deviation using the information stored in the files info[k] and the formulas given in [STB].
  For this we calculate for each $(j, h_j, T^{h_j}(N^{(j)}); i, h_i, T^{h_i}(N^{(i)}))$ the least positive value b such that

$$\text{dist}(T^{h_j+b}(N^{(j)}), T^{h_i+b}(N^{(i)})) \ge d \, .$$

In order not to duplicate information, once the value b corresponding to $(j, h_j, T^{h_j}(N^{(j)}); i, h_i, T^{h_i}(N^{(i)}))$ is calculated, we discard in this step the records $(j, h'_j, T^{h'_j}(N^{(j)}); i, h'_i, T^{h'_i}(N^{(i)}))$ such that $h_j + b \ge h'_j$ or $h_i + b \ge h'_i$ since these should have been taken into account in the previous step.

Remark 6.1. *Although we have not taken care of the possibility that $T^{20000}(N^{(i)}) = T^{20000}(N^{(j)})$ with $i \ne j$, this (very rare) possibility did not occurred in any of the simulations we have done. Moreover, in accordance with Proposition 4.3, in all these simulations, in particular in algorithm "entropia3", we always obtain that if $N \ne N'$ then $T^2(N) \ne T^2(N')$, so that the conjecture that $W^u(p)$ is dense in Λ is not contradicted.*

6.3 Appendix C: Pseudo-code of the algorithms employed

Here we give the pseudo code of the programs in a language close to FreePascal, the style of programming is procedural.

constants used

A0= 0.18; p = 100; A1 = 2; gamma=8.25; m0=50; rho=0.30; pipa=1024; na=400;

type of data structures used is standard, in particular "extended" means a floating point number of 10 bytes and "longint" or "integer" means an integer number occupying 4 bytes of memory according to the standards of IEEE. We also use arrays of extended or of integer and store the data in sequential files of records.

function S(h:integer):extended;
{INPUT: $h \in \mathbb{Z}$ OUTPUT: $S(h) \in \mathbb{R}^+$}
begin
if $(h < 0)$ or $(h > A1 * p)$ then S:=0   else S:= $1 - h/(A1 * p + 1)$
end;

function mrho(h:integer):extended;
{ INPUT: $h \in \mathbb{Z}$ OUTPUT: $m_\rho(h) = \left[\begin{array}{l} 1 \text{ if } 0 \leq h \mod 1 < \rho \\ 0 \text{ elsewhere} \end{array} \right.$ }
begin
entrho:=trunc(rho*p);
if ((h mod p) <entrho) then mrho:=0 else mrho:=1;
end;

function eme(N:extended):extended;
{INPUT: $N \in \mathbb{R}^+$ OUTPUT: $m(N) \in \mathbb{R}^+$}
begin
eme:=m0; lm:=N;
if lm>1 then eme:=eme*lm**(-gamma)
end;

procedure comienzoazar;
begin
randomize; semilla:=maxlongint;
end;

procedure AZAR(var n:longint);
{INPUT: random_seed OUTPUT: pseudo-random number$\in \mathbb{N}$}
begin
x:=random(200000); n:=x;
end;

function calculo(t:integer;ene:especial):extended;
{INPUT: $t \in \mathbb{Z}$ OUTPUT: $N \in \mathbb{R}^{2A_1p+1}$}
type
especial = array[1..2*A1*p+1] of extended;
begin
lc:=0;
for h:=floor(A0*p) to A1*p do begin
lc:=lc+ene[t-h]*eme(ene[t-h])*mrho(t-h)*S(h)   end;
calculo:=lc/p
end;

procedure eneinicial;
{INPUT: random;   OUTPUT: first vector $N \in \mathbb{R}^{A_1p+1}$}
begin
for i:=1 to A1*p do begin nhi[i]:=0; rnhi[i]:=0 end;
for i:= 1 to A1*p do begin
AZAR(l); nhi[i]:=l +500; { we assume that at least 500 rodents are alive}
rnhi[i]:=nhi[i]/55000 {we normalize values; $N_t := 1$ means 55000 rodents}

```
                           end;
            for i:=1 to A1*p do rnhaux[i]:=rnhi[i];
          for i:=A1*p+1 to 2*A1*p+1 do rnhaux[i]:=0;
               rnhi[A1*p+1]:=calculo(A1*p+1,rnhaux);
{warning: the coordinates of the vector N begin with 1 and finishes with A1*p+1}
                           end;

          procedure rnhgen(ene:rentrada;var ere:rentrada);
             {INPUT: N ∈ IR^{A1p+1}   OUTPUT: T²(N) ∈ IR^{A1p+1}}
                           type
                 rentrada = array[1..A1*p+1] of extended;
                          begin
                    t:=1;bo:=A1*p+1;
    for j:= 1 to bo do begin rnhaux[j]:=ene[j]; ere[j]:=0 end;
           for j:=bo+1 to 2*A1*p+1 do rnhaux[j]:=0;
    for t:=bo+1 to 2*A1*p+1 do begin z:=calculo(t,rnhaux);
               rnhaux[t]:=rnhaux[t]+z end;
          for i:=1 to bo do ere[i]:=rnhaux[i+A1*p]
                           end;

             begin {of program "ratones"}
            {INPUT: parameter values, random data
OUTPUT: na files of data representing time series of population of Microtus Epiroticus }
                   for jj:=1 to na do
                          begin
          rewrite(datos[jj]); comienzoazar;
          writeln('generating datos[',jj,']');
              eneinicial; rnhgen(rnhi,rnh);
              for j:=1 to 10000 do begin
            rnhv:=rnh; rnhgen(rnhv,rnh)
        end; {20000 iterates of T: N–¿T**(20000)(N)}
                 for i:=1 to pipa do begin
                    archi[i].numero:=0;
          for j:=1 to A1*p+1 do archi[i].serie[j]:=0;
                          end;
   archi[1].serie:=rnhi; archi[2].numero:=20000; archi[2].serie:=rnh;
                 for i:=3 to pipa do begin
       rnhv:=rnh; rnhgen(rnhv,rnh); {2 iterates of T each time}
         archi[i].numero:=20000+2*(i-2); archi[i].serie:=rnh;
                          end;
     for i:=1 to pipa do begin write(datos[jj],archi[i]); end;
                     end; {of "for jj"}
          writeln('type any key to finish'); ch:= readkey; exit
                   end. {of "ratones"}


   ——————————————————————————-


          procedure AZAR1(n: extended);
     {INPUT: random_seed   OUTPUT: pseudo-random number∈ IR}
                          begin
                x:=random; n:=x-0.5;
                  end; {of AZAR1}

               procedure eneperturb1;
    {INPUT: rnhi ∈ IR^{A1p+1}   OUTPUT: rnhi + Δrnhi ∈ IR^{A1p+1}}
                          begin
          for i:=1 to A1*p do begin nhi[i]:=0; end;
              for i:= 1 to A1*p do begin
```

```
                    AZAR1(l); nhi[i]:=l ;
                 rnhi[i]:=rnhi[i]+nhi[i]/(2**50)
                            end;
           for i:=1 to A1*p do rnhaux[i]:=rnhi[i];
        for i:=A1*p+1 to 2*A1*p+1 do rnhaux[i]:=0;
           rnhi[A1*p+1]:=calculo(A1*p+1,rnhaux);
                            end;
```

begin **{of program "ratones1"}**
{INPUT: a file "datos" generated by "ratones"
OUTPUT: a file "datosp" representing an initial small perturbation of "datos"}

```
                     for ii:=1 to na do
                            begin
             rewrite(datosp[ii]); comienzoazar;
         reset(datos[ii]); xx.numero:=-1; ayuda:=true;
          while (not Eof(datos)) and (ayuda=true) do
                            begin
                        read(datos,xx);
    if xx.numero=0 then begin archi[1].numero:=0; archi[1].serie:=xx.serie end;
         write(xx.numero,' serie ',xx.serie[1],' — ', xx.serie[100]);
                          writeln;
               if xx.numero=20000 then begin
                rnhi:= xx.serie; ayuda:=false
                            end;
                            end;
                 eneperturb1; rnh:=rnhi;
                     for i:=2 to pipa do
                            begin
                     archi[i].numero:=0;
          for j:=1 to A1*p+1 do archi[i].serie[j]:=0; end;
           archi[2].numero:=20000; archi[2].serie:=rnh;
                     for i:=3 to pipa do
                            begin
                      for j:=1 to 1 do
                            begin
                rnhv:=rnh; rnhgen(rnhv,rnh);
                            end;
        archi[i].numero:=20000+2*(i-2); archi[i].serie:=rnh;
                            end;
          for i:=1 to pipa do begin write(datosp,archi[i]); end;
                  reset(datos); reset(datosp);
            while (not Eof(datos)) and (not Eof(datosp)) do
                            begin
               read(datos,xx); read(datosp,yy);
  write(xx.numero,' serie ',xx.serie[1],' — ',yy.numero,' serie ', yy.serie[1]);
                          writeln;
                         end end;
           writeln('press any key to finish'); ch:= readkey; exit
                  end. {of "ratones1"}
```

---

function comparar(rnhx,rnhy: rentrada):longint;
{INPUT: $rnhx, rnhy \in I\!\!R^{A_1p+1}$ OUTPUT: 0 or 1}
{if "comparar" =0 then $||rnhx - rnhy|| < tol$, if 1 then > 0 }
```
                          begin
```

```
                              i:=1;
cmaux:=0; {we assume that at the beginning "comparar" is 0}
              while (cmaux=0) and (i<=A1*p) do
                             begin
        if (abs(rnhx[i]-rnhy[i])>=tol) then cmaux:=1; i:=i+1;
                             end;
                       comparar:=cmaux;
                             end;
```

begin **{of program "entropia3"}**
{INPUT: *na* files generated by "ratones",
OUTPUT: 16 files with pairs of time series d_i-near, $i = 1, 2, \ldots 16$; mean value $< N >$ of N_t;
absolute standard deviation of N_t}

```
                   for jj:=1 to na do begin
                  reset(datos[jj]); j:=0; z[jj]:=0;
                 while (not Eof(datos[jj])) do begin
                        for i:=1 to 8 do begin
                    y:=0; read(datos[jj],xx); j:=j+1;
                  for h:=1 to A1*p do y:=y+xx.serie[h];
                  y:=y/(A1*p); z[jj]:=z[jj]+y;        end
                      end; {of "while not Eof"}
                          z[jj]:=z[jj]/j;
        writeln('the mean value of file datos[',jj,'] is: ', z[jj]);
                        end; {of " for jj"}
            prom:=0; for jj:=1 to na do prom:= prom+z[jj];
          prom:=prom/na; writeln('total mean value ', prom);
            writeln('press any key to continue'); readkey(leer);
                   for jj:=1 to na do begin
                  reset(datos[jj]); j:=0; w[jj]:=0;
                 while (not Eof(datos[jj])) do begin
                             y:=0;
                    read(datos[jj],xx); j:=j+1;
          for h:=1 to A1*p do y:=y+abs(xx.serie[h]-z[jj]);
                y:=y/(A1*p); w[jj]:=w[jj]+y;
                        end; {of "while"}
                          w[jj]:=w[jj]/j;
     writeln('the absolute deviation value for datos[',jj,'] is ', w[jj]);
                        end; {of "for jj"}
                             dis:=0;
             for jj:=1 to na do dis:=dis+w[jj]; dis:=dis/na;
                  writeln('total deviation = ', dis);
             writeln('to continue press ENTER'); readln(leer);
                         {we collect data}
                        tol:=1/(2**(10));
```
{"tol" is what is called d in the algorithm; here we exemplify with $tol \approx 0.001$ }
```
                          rewrite(info);
                    for jj:=1 to na-1 do begin
                        reset(datos[jj]);
     If (not Eof(datos[jj])) then read(datos[jj],xx); {we discard the first}
                     for ii:=jj+1 to na do begin
                 while (not Eof(datos[jj])) do begin
                 read(datos[jj],xx); reset(datos[ii]);
     If (not Eof(datos[ii])) then read(datos[ii],yy); {we discard the first}
                 while (not Eof(datos[ii])) do begin
                        read(datos[ii],yy);
                 u:=comparar(xx.serie,yy.serie);
```
             if u=0 then {that is: $||T^l(N) - T^l(N')|| < tol$}

```
begin
estx.numarch1:=jj; estx.numarch2:=ii;
estx.numiter1:=xx.numero; estx.numiter2:=yy.numero;
estx.punto1:=xx.serie; estx.punto2:=yy.serie;
write(info,estx);
end {of "if"}
end {of "while not Eof(datos[ii])"}
end {of "while not Eof(datos[jj])"}
end {of "for ii"} end {of "for jj"}
writeln('teclee cualquier tecla para finalizar'); ch:= readkey; exit;
end. {of program "entropia"}
```

───────────────────────────────────────

```
begin {of program "entropia4"}
{INPUT: A file with pairs of time series $d_i$-near,
```
OUTPUT: an estimation of the second order Kolmogorov-entropy $\widetilde{K}$;
an estimation of its standard deviation $\sigma_{\widetilde{K}}$}
```
rewrite(androide);
for jj:=1 to na do reset(datos[jj]);
tol:=1/(2**(10))
begin
base:=1; reset(info);
while (not Eof(info)) do begin
read(info,estx);
if base>tope then begin writeln('error, table too small'); halt end;
tabla[base].numarch1:=estx.numarch1; tabla[base].numarch2:=estx.numarch2;
tabla[base].numiter1:=estx.numiter1; tabla[base].numiter2:=estx.numiter2;
rnh1:=estx.punto1; rnh2:=estx.punto2; j:=0;
repeat
rnh1v:=rnh1; rnhgen(rnh1v,rnh1); rnh2v:=rnh2; rnhgen(rnh2v,rnh2); j:=j+1;
until comparar(rnh1,rnh2)<>0;
tabentr[base]:=j;
if base>1 then begin
if (tabla[base].numarch2=tabla[base-1].numarch2) and
(tabla[base-1].numiter2+tabentr[base-1]>=tabla[base].numiter2)
then base:=base-1 {overlap of data}
else
begin
if (tabla[base].numarch1=tabla[base-1].numarch1) and
(tabla[base-1].numiter1+tabentr[base-1]>=tabla[base].numiter1)
then base:=base-1 {overlap of data}
end end;
base:=base+1;
end; {of "while not Eof"}
tiempos:=0;
for i:=1 to base-1 do begin
tiempos:=tiempos+tabentr[i]; end;
tiempos:=tiempos/(base-1); entropy:=-Ln(abs(1-1/tiempos));
writeln('the values of $b_j$ are');
for j:=1 to base -1 do
begin writeln('b',j,' = ',tabentr[j],' | '); end;
writeln('average of $b_j$ is <b > = ',tiempos);
writeln(' Entropy estimated is ',entropy, ', the size of the sample is ',base-1);
rna:=base-1;
writeln('standard deviation of K is :', 1/(sqrt(rna)*entropy*sqrt(tiempos*(tiempos-1))));
```

resultado:=entropy;
desvio:= 1/(sqrt(rna)*entropy*sqrt(tiempos*(tiempos-1)));
writeln('to finish press any key '); ch:=readkey;
end. {of program "entropia4"}

6.4   Appendix D: pseudo-code of homclin4.

Program "homclin4"
{INPUT: a table with the candidates to be homoclinic points}
{OUTPUT: A point in $W^u_{loc}(p)$ such that near it there is numerical evidence that it exists a homoclinic point}
This program uses, apart from the functions and procedures defined above, two functions "distl2" and "angulo".
"distl2" computes the Euclidean distance between points, and "angulo" computes the angle between a pair of
vectors. "angulo" uses a function "prodint" that calculates the inner product between vectors. The program
also uses two procedures, "minimo" that computes the minimum between real data stored in a file called
"candihomclin" and "iterar" that iterates the function T^2 a prescribed number of times.

function distl2(rnhx,rnhy: rentrada):extended;
{calculates euclidean distance between points}
var cmaux,i:longint; raux,dist:extended; maximo:extended; rnhd:rentrada;
begin
i:=2; maximo:=abs(rnhx[1]-rnhy[1]);
while i<=A1*p+1 do begin
if abs(rnhx[i]-rnhy[i])> maximo then maximo:=abs(rnhx[i]-rnhy[i]);
i:=i+1   end;
if maximo<>0 then
for i:=1 to A1*p+1 do rnhd[i]:= abs(rnhx[i]-rnhy[i])/maximo;
i:=1; dist:=0; { assume distance is 0}
if maximo<>0 then
while (i<=A1*p+1) do begin
dist:=dist+rnhd[i]*rnhd[i]; i:=i+1;   end;
distl2:=maximo*sqrt(dist);
end;

function prodint(rnhx,rnhy: rentrada):extended;
{computes inner product of vectors}
var i:longint; prod:extended; rnhd,rnhe:rentrada;maximox,maximoy:extended;
begin
i:=2; maximox:=abs(rnhx[1]);maximoy:=abs(rnhy[1]);
while i<=A1*p+1 do begin
if abs(rnhx[i]) > maximox then maximox:=abs(rnhx[i]);
if abs(rnhy[i]) > maximoy then maximoy:=abs(rnhy[i]);
i:=i+1   end;
if (maximox*maximoy<>0) then begin
for i:=1 to A1*p+1 do
begin rnhd[i]:= rnhx[i]/maximox; rnhe[i]:=rnhy[i]/maximoy end;
i:=1; prod:=0;
while i<=A1*p+1 do begin
prod:=prod+rnhd[i]*rnhe[i]; i:=i+1   end;
prodint:=prod*maximox*maximoy;   end
else prodint:=0;
end;

function angulo(rnhx,rnhy:rentrada):extended;
var equis, ye, zeta:extended;
begin
zeta:= prodint(rnhx,rnhy); equis:=sqrt(prodint(rnhx,rnhx)); ye:=sqrt(prodint(rnhy,rnhy));
if (equis=0) or (ye=0) then angulo:=0 else angulo:=arccos(zeta/(equis*ye));
end;

procedure minimo;
var minaux:extended;fijmin:rentrada; seguir:boolean;
begin
min:=1; seguir:=true; {"min" is set to a value which will not be the minimum}
while (not Eof(refcandihomclin)) and (seguir=true) do begin
read(refcandihomclin,homocl); if homocl.punto[2]<>0 then begin
seguir:=false; min:=10**(-2); refhomocl:=homocl   end;
if (homocl.punto[2]=0) and (homocl.punto[1]<min) and (homocl.numh<700)
and (homocl.numh>10) then begin
min:=homocl.punto[1]; refhomocl:=homocl   end;
end; {of while}
end;

procedure iterar(paso:integer;sota:rentrada;var sota1:rentrada);
var rnhj,rnhjv:rentrada; {"paso" controls the number of iterations}
begin rnhjv:=sota;
for j:=1 to paso do
begin rnhgen(rnhjv,rnhj); rnhjv:=rnhj;   end;
sota1:=rnhj;
end;

begin {of homclin4}
while not Eof(candihomclin) do begin
read(candihomclin,homocl);
if (homocl.punto[2]=0.0) and (homocl.punto[1]¡0.001) then
write(refcandihomclin,homocl);
if homocl.punto[2]¡¿0.0 then write(refcandihomclin,homocl)
end; {of while}
reset(refcandihomclin);
minimo;
writeln('minimum distance to p is ', min);
writeln('value of i=',refhomocl.numi,' iterate closest to p is ',refhomocl.numh+10);
writeln('initial approximation to candidate to homoclinic point M is ');
for j:=1 to A1*p+1 do begin
fijo12[j]:=fijo6[j]+(10000-refhomocl.numi)*fijo8[j];
if (j mod 3=0) then writeln(fijo12[j],'—')
else write(fijo12[j],' —');   end;
tolerancia: {a label of reference}
if (refhomocl.numh mod 2 = 0) then techo:=refhomocl.numh+10
else techo:=refhomocl.numh+10;
fijo8:=restar(fijo6,fijo4);
for j:=1 to A1*p+1 do fijo8[j]:=fijo8[j]/10000;
writeln;
writeln('Next we refine the choice, in particular we find L and R');
writeln('points in $[T^{38}(p), T^{40}(p)]$ identified with $W^u_{loc}(p)$');
writeln('such that M is between them and such that the iterates');
writeln('of L and R are in different components with respect to');
writeln('the local stable manifold $W^s_{loc}(p)$ of p.');
writeln('For convenience we continue to denote by M, L and R their iterates by T^2');
writeln('Enter gap distance as a real exponent of 2 not greater than 30');
writeln('the gap distance will be $2^{(-exponent)}$');
write('To finish the program enter exponent=0, exponente = ');
readln(semillon);
if semillon< 0 then
begin semillon:=-semillon;
writeln('a negative value has been entered, ',semillon,' will be assumed');
end;

```
                            if semillon> 20 then
                                  begin
          writeln('exponent too large, a value of 10 will be assumed');
                             semillon:=10 end;
                            while semillon<> 0 do
                                  begin
                   tol:=2**(semillon); writeln('tol=',1/tol);
                             for j:=1 to A1*p+1 do
                                  begin
              fijo12[j]:=fijo6[j]+(10000-refhomocl.numi)*fijo8[j];
                         fijo11[j]:=fijo12[j]-fijo8[j]/tol;
                         fijo13[j]:=fijo12[j]+fijo8[j]/tol;
                       fijo115[j]:=fijo12[j]-fijo8[j]/(2*tol);
                      fijo135[j]:=fijo12[j]+fijo8[j]/(2*tol);
                                   end;
                              while techo> 0 do
                                  begin
                         if techo>= 10 then begin
                iterar(10,fijo12,rnh1); iterar(10,fijo11,rnh0);
                iterar(10,fijo13,rnh2); iterar(10,fijo115,rnh05);
                 iterar(10,fijo135,rnh25); techo:=techo-10;
       writeln('distance between left and right iterates L and R is ',distl2(rnh0,rnh2));
            writeln('dist(L,L1)+dist(L1,M)+dist(M,R1)+dist(R1,R)= ',
       distl2(rnh0,rnh05)+distl2(rnh05,rnh1)+distl2(rnh1,rnh25)+distl2(rnh25,rnh2));
                       if (distl2(rnh0,rnh2) > 0.0001) or
       (distl2(rnh0,rnh05)+distl2(rnh05,rnh1)+distl2(rnh1,rnh25)+distl2(rnh25,rnh2)
                         > 1.001*distl2(rnh0,rnh2))
                                 then begin
          writeln('distance between iterates is too large or curvature is big');
                      techo:=techo+10; iterar(8,fijo12,rnh1);
                   iterar(8,fijo11,rnh0); iterar(8,fijo13,rnh2);
                  iterar(8,fijo115,rnh05); iterar(8,fijo135,rnh25);
                               techo:=techo-8;
      writeln('iterating 8 times the new distance between L and R is ',distl2(rnh0,rnh2));
            writeln('dist(L,L1)+dist(L1,M)+dist(M,R1)+dist(R1,R)= ',
       distl2(rnh0,rnh05)+distl2(rnh05,rnh1)+distl2(rnh1,rnh25)+distl2(rnh25,rnh2));
                       if (distl2(rnh0,rnh2) > 0.0001) or
       (distl2(rnh0,rnh05)+distl2(rnh05,rnh1)+distl2(rnh1,rnh25)+distl2(rnh25,rnh2)
                         > 1.001*distl2(rnh0,rnh2))
                                 then begin
       writeln('distance between iterates continues to be too large or curvature is big');
                      techo:=techo+8; iterar(2,fijo12,rnh1);
                   iterar(2,fijo11,rnh0); iterar(2,fijo13,rnh2);
                  iterar(2,fijo115,rnh05); iterar(2,fijo135,rnh25);
                               techo:=techo-2;
      writeln('iterating 2 times the new distance between L and R is ',distl2(rnh0,rnh2));
            writeln('dist(L,L1)+dist(L1,M)+dist(M,R1)+dist(R1,R)= ',
       distl2(rnh0,rnh05)+distl2(rnh05,rnh1)+distl2(rnh1,rnh25)+distl2(rnh25,rnh2));
                         end {of inner "if then"}
                       end end {of outer "if then"}
                 else begin { now "techo" is less or equal than 10}
                iterar(techo,fijo12,rnh1); iterar(techo,fijo11,rnh0);
                   iterar(techo,fijo13,rnh2); techo:=0
                                   end;
                   fijo8:=restar(rnh2,rnh0); fijo12:=rnh1;
                           for j:=1 to A1*p+1 do
```

begin

fijo11[j]:=fijo12[j]-fijo8[j]/tol; fijo13[j]:=fijo12[j]+fijo8[j]/tol;

fijo115[j]:=fijo12[j]-fijo8[j]/(2*tol); fijo135[j]:=fijo12[j]+fijo8[j]/(2*tol);

end;

writeln(' iterates= ',refhomocl.numh+10-techo,'

distance between endpoints L and R previous to iteration is ');

writeln(distl2(fijo11,fijo13));

if distl2(fijo11,fijo13) ¿ 0.00001 then for j:=1 to A1*p+1 do

begin

fijo11[j]:=fijo12[j]-fijo8[j]/(16*tol); fijo13[j]:=fijo12[j]+fijo8[j]/(16*tol)

end;

if distl2(fijo11,fijo13)¡=0.0000000000000001 then begin

writeln('tol is too small, please, reduce the exponent'); goto tolerancia; end;

end; {del while techo}

dist0:=distl2(fijo,rnh0); dist1:=distl2(fijo,rnh1);

dist2:=distl2(fijo,rnh2); writeln;

writeln(' Euclidean dist from p to original point T^2,'*(refhomocl.numh+10),'is ',dist1);

writeln(' Euclidean dist from p to left point ',dist0);

writeln(' Euclidean dist from p to right point ',dist2);

writeln(' Euclidean dist between left and right points is '); writeln(distl2(rnh0,rnh2));

if (refhomocl.numh mod 2 = 0) then techo:=refhomocl.numh+10

else techo:=refhomocl.numh+10;

fijo14:=restar(fijo4,fijo2); fijo16:=restar(rnh2,rnh0); rnhgen(fijo6,fijo8);

writeln('angle between $W_e^u(p)$ and iterated arc LM is = ');

write(angulo(fijo14,fijo16));

writeln(' angle in degrees is approx = ' ,round(angulo(fijo14,fijo16)*180/Pi));

writeln(' Euclidean dist between left end-point of $W_e^u(p)$ and L is ');

writeln(distl2(fijo6,rnh0));

writeln(' Euclidean dist between left end-point of $W_e^u(p)$ and R is ');

writeln(distl2(fijo6,rnh2));

writeln(' Euclidean dist between right end-point of $W_e^u(p)$ and L is ');

writeln(distl2(fijo8,rnh0));

writeln(' Euclidean dist between right end-point of $W_e^u(p)$ and R is ');

writeln(distl2(fijo8,rnh2));

rnhgen(rnh0,rnh0v);rnhgen(rnh2,rnh2v);

writeln('rate of dist between rnh0, rnh2 and their iterates by T^2 is ');

writeln(distl2(rnh0v,rnh2v)/distl2(rnh0,rnh2));

fijo18:=restar(fijo,rnh0); fijo20:=restar(fijo,rnh2);

writeln('angle between vectors (p,L) and (p,R) is ');

write(angulo(fijo18,fijo20));

writeln(' angle in degrees is approx = ',round(angulo(fijo18,fijo20)*180/Pi));

fijo18:=restar(fijo,rnh0v); fijo20:=restar(fijo,rnh2v);

writeln('angle between vectors (p,$T^2(L)$) and (p,$T^2(R)$) is ');

write(angulo(fijo18,fijo20));

writeln(' angle in degrees is approx = ',round(angulo(fijo18,fijo20)*180/Pi));

for ii:=1 to 6 do begin

newfix[2*ii-1]:=rnh0v;newfix[2*ii]:=rnh2v;

rnhgen(newfix[2*ii-1],rnh0v);rnhgen(newfix[2*ii],rnh2v);

fijo18:=restar(fijo,rnh0v); fijo20:=restar(fijo,rnh2v);

writeln('angle between vectors (p,T^2,*(ii+1),'(L)) and (p,T^2,*(ii+1),'(R)) is ');

write(angulo(fijo18,fijo20));

writeln(' angle in degrees is approx = ',round(angulo(fijo18,fijo20)*180/Pi));

end; writeln;

nuevofijo2:=restar(nuevofijo,rnh0); nuevofijo4:=restar(nuevofijo,rnh2);

writeln('angle between vectors $(\hat{p}, L)$ and $(\hat{p}, R)$ is ');

write(angulo(nuevofijo2,nuevofijo4));

writeln(' angle in degrees is approx = ',round(angulo(nuevofijo2,nuevofijo4)*180/Pi));
writeln; fijo8:=restar(fijo6,fijo4);
for j:=1 to A1*p+1 do fijo8[j]:=fijo8[j]/10000;
writeln; writeln('Enter gap distance as a real exponent of 2 no greater than 30');
writeln('last exponent used is ',semillon);
write('To finish the program enter exponent=0, exponent = ');
readln(semillon);
if semillon< 0 then begin
semillon:=-semillon;
writeln('a negative value has been entered, ',semillon,' will be assumed');
end;
if semillon> 20 then begin
writeln('exponent too large, a value of 10 will be assumed');
semillon:=10;
end end;
write(candihomclin2,homocl);
writeln('To continue press ENTER'); read(leer); writeln('Press any key to finish'); ch:=readkey;
end.

6.5   Appendix E: coordinates of fixed point.

Approximate coordinates of the fixed point $p \in \mathbb{R}^{201}$ of $T^2(N)$ are given in the following table.

$1.2326490487970465E + 0000$	$1.2110482116814741E + 0000$	$1.1906886685005045E + 0000$
$1.1717713776064593E + 0000$	$1.1545319083055524E + 0000$	$1.1392463406313234E + 0000$
$1.1262379872047533E + 0000$	$1.1158849347082125E + 0000$	$1.1086283238645345E + 0000$
$1.1049811591622351E + 0000$	$1.1055372415914759E + 0000$	$1.1109795220165019E + 0000$
$1.1220867468556205E + 0000$	$1.1397366769671205E + 0000$	$1.1649033774342367E + 0000$
$1.1986450973703732E + 0000$	$1.2420781384673748E + 0000$	$1.2087810376155805E + 0000$
$1.1754839367637863E + 0000$	$1.1421868359119921E + 0000$	$1.1088897350601978E + 0000$
$1.0755926342084036E + 0000$	$1.0422955333566094E + 0000$	$1.0089984325048151E + 0000$
$9.7570133165302088E - 0001$	$9.4240423080122665E - 0001$	$9.0910712994943241E - 0001$
$8.7581002909763817E - 0001$	$8.4251292824584393E - 0001$	$8.0921582739404970E - 0001$
$7.7591872654225546E - 0001$	$7.4463460038547187E - 0001$	$7.1528062031461468E - 0001$
$6.8777896512033852E - 0001$	$6.6205661495172081E - 0001$	$6.3804515773116380E - 0001$
$6.1568060751297437E - 0001$	$5.9490323430406656E - 0001$	$5.7565740489494235E - 0001$
$5.5789143427761938E - 0001$	$5.4155744725456250E - 0001$	$5.26611124986901219E - 0001$
$5.1301221031245340E - 0001$	$5.0072314898939969E - 0001$	$4.8971023744324276E - 0001$
$4.7994290586969646E - 0001$	$4.7139375896640730E - 0001$	$8.3241286907550863E - 0001$
$1.1774339903181779E + 0000$	$1.5083543629435397E + 0000$	$1.8248209517576603E + 0000$
$2.1271151896510347E + 0000$	$2.4160208699409385E + 0000$	$2.6923595086534992E + 0000$
$2.9569342453691580E + 0000$	$3.2105209755162044E + 0000$	$3.4538676455028848E + 0000$
$3.6876953663770639E + 0000$	$3.9126999878684479E + 0000$	$4.1295537791098290E + 0000$
$4.3389071105434127E + 0000$	$4.5413901032229220E + 0000$	$4.7376142351332896E + 0000$
$4.9281739025629719E + 0000$	$5.1136479378130289E + 0000$	$5.4622938956939894E + 0000$
$5.5692297788707215E + 0000$	$5.5592343648444013E + 0000$	$5.5318006317643429E + 0000$
$5.5004355530098589E + 0000$	$5.4679067992365053E + 0000$	$5.4349605692581039E + 0000$
$5.4018415509009528E + 0000$	$5.3686426726693849E + 0000$	$5.3354035019177187E + 0000$
$5.3021425008934430E + 0000$	$5.2688689546762325E + 0000$	$5.2355878340449946E + 0000$
$5.2023019446101286E + 0000$	$5.1690129433312447E + 0000$	$5.1357218479879554E + 0000$
$5.1024293056141891E + 0000$	$5.0691357402363859E + 0000$	$5.0358409167684584E + 0000$
$5.0025458322320211E + 0000$	$4.9692507815459950E + 0000$	$4.9359558081745136E + 0000$
$4.9026609256467161E + 0000$	$4.8693661415807602E + 0000$	$4.8360714621823154E + 0000$
$4.8027768933916233E + 0000$	$4.7694824412520072E + 0000$	$4.7361881120565536E + 0000$
$4.7028939124217354E + 0000$	$4.6695998493345792E + 0000$	$4.6363059301899531E + 0000$
$4.6030121628245493E + 0000$	$4.5697185555504526E + 0000$	$4.5364251171897114E + 0000$
$4.5031318571107021E + 0000$	$4.4698387852668010E + 0000$	$4.4365458667634358E + 0000$

$4.4032531103361234E + 0000$	$4.3699605252784317E + 0000$	$4.3366681214830976E + 0000$
$4.3033759094851984E + 0000$	$4.2700839005086080E + 0000$	$4.2367921065161327E + 0000$
$4.2035005402636639E + 0000$	$4.1702092153587075E + 0000$	$4.1369181463236780E + 0000$
$4.1036273486643819E + 0000$	$4.0703368389441578E + 0000$	$4.0370466348641843E + 0000$
$4.0037567553505172E + 0000$	$3.9704672206484750E + 0000$	$3.9371780524250502E + 0000$
$3.9038809515732566E + 0000$	$3.8705838507214630E + 0000$	$3.8372867498696694E + 0000$
$3.8039896490178758E + 0000$	$3.7706925481660822E + 0000$	$3.7373954473142887E + 0000$
$3.7040983464624951E + 0000$	$3.6708012456107015E + 0000$	$3.6375041447589079E + 0000$
$3.6042070439071143E + 0000$	$3.5709099430553207E + 0000$	$3.5376128422035272E + 0000$
$3.5043157413517336E + 0000$	$3.4710186404999400E + 0000$	$3.4377218198045367E + 0000$
$3.4044252993439741E + 0000$	$3.3711291008502547E + 0000$	$3.3378332478631211E + 0000$
$3.3045377659003338E + 0000$	$3.2712426826458989E + 0000$	$3.2379480281583430E + 0000$
$3.2046538351014030E + 0000$	$3.1713601389998044E + 0000$	$3.1380669785231534E + 0000$
$3.1047743958013707E + 0000$	$3.0714824367755503E + 0000$	$3.0381911515886585E + 0000$
$3.0049005950210884E + 0000$	$2.9716108269767829E + 0000$	$2.9383219130264393E + 0000$
$2.9050851936346499E + 0000$	$2.8718528731824328E + 0000$	$2.8386253205708071E + 0000$
$2.8054029398018444E + 0000$	$2.7721861737242195E + 0000$	$2.7389755082227071E + 0000$
$2.7057714769095989E + 0000$	$2.6725746663842937E + 0000$	$2.6393857221368817E + 0000$
$2.6062053551826095E + 0000$	$2.5730343495269334E + 0000$	$2.5398735705757476E + 0000$
$2.5067239746226584E + 0000$	$2.4735866195652930E + 0000$	$2.4404626770260829E + 0000$
$2.4073534460803303E + 0000$	$2.3742603688263723E + 0000$	$2.3411850480701324E + 0000$
$2.3081292418906175E + 0000$	$2.2750949308401385E + 0000$	$2.2420843237904744E + 0000$
$2.2090998874484834E + 0000$	$2.1761443801015639E + 0000$	$2.1432208902575048E + 0000$
$2.1103328809563707E + 0000$	$2.0774842406657146E + 0000$	$2.0446793418285290E + 0000$
$2.0119231083205840E + 0000$	$1.9792210932957181E + 0000$	$1.9465795691608410E + 0000$
$1.9140056317346885E + 0000$	$1.8815073210149785E + 0000$	$1.8490937614183503E + 0000$
$1.8167753248789589E + 0000$	$1.7845638208093313E + 0000$	$1.7524727176575964E + 0000$
$1.7205174017417589E + 0000$	$1.6887154797718421E + 0000$	$1.6570871330214122E + 0000$
$1.6256555322735251E + 0000$	$1.5944473242987124E + 0000$	$1.5634932024257111E + 0000$
$1.5328285757821936E + 0000$	$1.5024943539853568E + 0000$	$1.4725378663866274E + 0000$
$1.4430139373005373E + 0000$	$1.4139861407724177E + 0000$	$1.3855282600450476E + 0000$
$1.3577259774985657E + 0000$	$1.3306788197951572E + 0000$	$1.3045023793744458E + 0000$
$1.2793308262228334E + 0000$	$1.2553197117722144E + 0000$	$1.2326490487971048E + 0000$

From the analytic expression of T, it is clear that $T(p) \neq p$ so that p has period 2.

6.6   Appendix F: homoclinic points search.

We plot a projection of the attractor in three-dimensional space (averaging some coordinates at the beginning of the year, in the middle of the year and in Spring). For that purpose we use MATLAB$^{\copyright}$. When we plot the image of the first 1000 iterates of a single point of the local unstable manifold $W_{loc}^u(p)$ we roughly recover the image of Λ obtained plotting all the sequences of points pseudo-randomly generated. This is an indication that $W^u(p)$ may be dense in Λ. The small red circle in the figures, indicates the approximate position of the fixed point p.

We also give approximate coordinates of the homoclinic point y_0 and angular values obtained with an initial length of I_0 of 1.122×10^{-7} in the appendix below with two copies of runnings of "homclin7.exe" which is a refinement of "homclin4.exe" which generates an output close to LaTeX. In these runs we use three values for the parameter "exponent", one of them is 10.80 and the other is 15.03. We also exhibit one exponent, 11.00 which fails to detect homoclinic points. Observe that 10.80 is not very far apart from 11.00.

We only exhibit samples of the runs since they are rather extensive. It is possible to observe that the program corrects the quantity of iterations when the results are larger than certain bounds.

Runs of "homoclin7"

Enter gap as an exponent of 2 not greater than 20 and greater than 3, we choose gap=$2^{exponent}$.
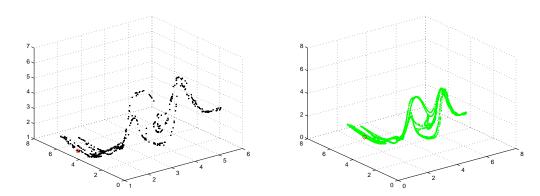This gap will be used to divide the distance between three consecutive points

Figure 2: A thousand iterates of $y \in W^u_{loc}(p)$ projected in $3D$ (in black) versus the projection of Λ (in green).

of the initial subdivision of $[T^{38}(p), T^{40}(p)]$ centered around
the rough homoclinic point y_0 previously found.
To finish the program enter exponent=0,
exponent = 1.0800000000000000E+0001 gap = 1782.8875536 exponent = 10.8000000
iter= 0, dist(L,R) previous to iteration is 3.5890677421214318E-0010
dist between L and R after applying T^{20} is 6.3294602762248157E-0007
dist(L,L1)+dist(L1,M)+dist(M,R1)+dist(R1,R)= 6.3294602764511862E-0007
iter= 10, dist(L,R) previous to iteration is 7.1002349673070908E-0010
dist between L and R after applying T^{20} is 1.0395709474479636E-0008
dist(L,L1)+dist(L1,M)+dist(M,R1)+dist(R1,R)= 1.0395709474479637E-0008
iter= 20, dist(L,R) previous to iteration is 1.1661654789708924E-0011
dist between L and R after applying T^{20} is 6.7123467133053106E-0010
dist(L,L1)+dist(L1,M)+dist(M,R1)+dist(R1,R)= 6.7123467133053267E-0010

$$\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots$$

iter= 600, dist(L,R) previous to iteration is 1.7874017013569691E-0011
dist between L and R after applying T^{20} is 3.5203152021803773E-0006
dist(L,L1)+dist(L1,M)+dist(M,R1)+dist(R1,R)= 3.5203152021804383E-0006
iter= 610, dist(L,R) previous to iteration is 3.9490041813784173E-0009
dist between L and R after applying T^{20} is 4.3191750633612666E-0005
dist(L,L1)+dist(L1,M)+dist(M,R1)+dist(R1,R)= 4.3191750633619148E-0005
iter= 620, dist(L,R) previous to iteration is 4.8451457912339789E-0008
dist between L and R after applying T^{18} is 2.7846424585034373E-0003
dist(L,L1)+dist(L1,M)+dist(M,R1)+dist(R1,R)= 2.7846424590282723E-0003
iter= 629, dist(L,R) previous to iteration is 3.1237443470096728E-0006

Sup distance from fixed point p to point $T^{1258}(y_0)$ is 0.00022627982
L1 distance from fixed point p to point $T^{1258}(y_0)$ is 0.01670265394
Euclidean distance from fixed point p to point $T^{1258}(y_0)$ is 0.00141044319
Euclidean distance from fixed point p to point L is 0.00007862956
Euclidean distance from fixed point p to point R is 0.00280177267
Euclidean distance between L and R is 0.00278464246
angle between $W^u_e(p)$ and iterated arc $LR = 0.00003$ radians, angle in degrees is ≈ 0
angle between vectors (p, L) and (p, R) is 1.33746 angle in degrees is ≈ 77
angle between vectors $(p, T^2(L))$ and $(p, T^2(R))$ is 3.01126 radians, angle in degrees is ≈ 173
angle between vectors $(p, T^4(L))$ and $(p, T^4(R))$ is 3.13900 radians angle in degrees is ≈ 180
angle between vectors $(p, T^6(L))$ and $(p, T^6(R))$ is 3.14031 radians angle in degrees is ≈ 180

Coordinates of initial approximation to homoclinic point y_0 are

1.232652605	1.211051279	1.190691205	1.171773334	1.154533228
1.139246960	1.126237833	1.115883924	1.108626365	1.104978147
1.105533061	1.110974047	1.122079841	1.139728196	1.164893175
1.198633030	1.242064077	1.208766906	1.175469736	1.142172565
1.108875395	1.075578224	1.042281054	1.008983883	0.975686713
0.942389542	0.909092372	0.875795201	0.842498031	0.809200860
0.775903690	0.744619505	0.715265478	0.687763786	0.662041411
0.638029941	0.615665389	0.594888025	0.575642216	0.557876277
0.541542333	0.526596191	0.512997217	0.500708233	0.489695409
0.479928177	0.471379141	0.832391559	1.177406005	1.508319728
1.824779645	2.127067226	2.415966274	2.692298311	2.956866483
3.210446691	3.453786886	3.687608184	3.912606440	4.129453928
4.338801024	4.541277853	4.737495899	4.928049563	5.113517681
5.462154712	5.569115435	5.559124568	5.531692464	5.500328368
5.467800406	5.434854900	5.401736578	5.368538383	5.335299888
5.302039560	5.268766684	5.235486232	5.202201011	5.168912678
5.135622250	5.102330375	5.069037476	5.035743319	5.002448902
4.969154518	4.935860211	4.902565995	4.869271877	4.835977865
4.802683963	4.769390177	4.736096515	4.702802982	4.669509585
4.636216333	4.602923232	4.569630292	4.536337520	4.503044927
4.469752522	4.436460270	4.403168180	4.369876262	4.336584525
4.303292980	4.270001638	4.236710511	4.203419612	4.170128954
4.136838552	4.103548422	4.070258579	4.036969042	4.003679830
3.970390963	3.937102462	3.903806027	3.870509592	3.837213158
3.803916723	3.770620289	3.737323854	3.704027419	3.670730985
3.637434550	3.604138116	3.570841681	3.537545246	3.504248812
3.470952377	3.437656223	3.404360368	3.371064836	3.337769649
3.304474833	3.271180416	3.237886428	3.204592901	3.171299871
3.138007377	3.104715460	3.071424167	3.038133548	3.004843657
2.971554555	2.938266307	2.905030261	2.871798613	2.838571735
2.805350028	2.772133937	2.738923947	2.705720592	2.672524458
2.639336191	2.606156503	2.572986176	2.539826078	2.506677163
2.473540491	2.440417232	2.407308686	2.374216295	2.341141663
2.308086547	2.275052928	2.242043015	2.209059275	2.176104467
2.143181680	2.110294376	2.077446445	2.044642259	2.011886743
1.979185450	1.946544654	1.913971450	1.881473880	1.849061068
1.816743387	1.784532649	1.752442321	1.720487792	1.688686670
1.657059138	1.625628368	1.594421010	1.563467758	1.532804025
1.502470723	1.472515186	1.442992241	1.413965467	1.385508654
1.357707488	1.330661505	1.304486305	1.279316068	1.255306356
1.232637195				

Euclidean distance from p to u_0 =   0.000894332

Figure 3: Coordinates of (rough) homoclinic point y_0 associated to p.

angle between vectors $(p, T^8(L))$ and $(p, T^8(R))$ is 3.13743 radians angle in degrees is ≈ 180
angle between vectors $(p, T^{10}(L))$ and $(p, T^{10}(R))$ is 3.12545 radians angle in degrees is ≈ 179
angle between vectors $(p, T^{12}(L))$ and $(p, T^{12}(R))$ is 3.10750 radians angle in degrees is ≈ 178
angle between vectors $(p, T^{14}(L))$ and $(p, T^{14}(R))$ is 3.05523 radians angle in degrees is ≈ 175

Enter gap distance as a real exponent of 2 between 3 and 20
last exponent used is 10.80000000
To finish the program enter exponent=0. Chosen exponent = 15.03200000
gap = 33502.9380910 exponent = 15.0320000
iter= 0, dist(L,R) previous to iteration is 1.9099531465388560E-0011
dist between L and R after applying T^{30} is 1.4460501331114946E-0008
dist(L,L1)+dist(L1,M)+dist(M,R1)+dist(R1,R)= 1.4460501331114946E-0008
iter= 15, dist(L,R) previous to iteration is 8.6323753141429872E-0013
dist between L and R after applying T^{30} is 1.6945248777241631E-0009
dist(L,L1)+dist(L1,M)+dist(M,R1)+dist(R1,R)= 1.6945248777241637E-0009
iter= 30, dist(L,R) previous to iteration is 1.0115684115597356E-0013
dist between L and R after applying T^{30} is 2.4222230946916936E-0009
dist(L,L1)+dist(L1,M)+dist(M,R1)+dist(R1,R)= 2.4222230946916942E-0009

$$\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots$$

iter= 594, dist(L,R) previous to iteration is 1.2923688152099415E-0009
dist between L and R after applying T^{30} is 9.5328543032789122E-0004
dist(L,L1)+dist(L1,M)+dist(M,R1)+dist(R1,R)= 9.5328543041795548E-0004
distance between iterates is too large or curvature is big
dist between L and R after applying T^{16} is 6.3306020056154999E-0008
dist(L,L1)+dist(L1,M)+dist(M,R1)+dist(R1,R)= 6.3306020056154999E-0008
iter= 602, dist(L,R) previous to iteration is 3.7791322627648531E-0012
dist between L and R after applying T^{30} is 1.9472974658090783E-0005
dist(L,L1)+dist(L1,M)+dist(M,R1)+dist(R1,R)= 1.9472974658261263E-0005
iter= 617, dist(L,R) previous to iteration is 1.1624636981219546E-0009
dist between L and R after applying T^{24}) is 2.7688866606693677E-0003
dist(L,L1)+dist(L1,M)+dist(M,R1)+dist(R1,R)= 2.7688866611853443E-0003
iter= 629, dist(L,R) previous to iteration is 1.6529216948955862E-0007

Sup distance from fixed point p to point $T^{1258}(y_0)$ is 0.00022627982
L1 distance from fixed point p to point $T^{1258}(y_0)$ is 0.01670265394
Euclidean distance from fixed point p to point $T^{1258}(y_0)$ is 0.00141044319
Euclidean distance from fixed point p to point L is 0.00279389720
Euclidean distance from fixed point p to point R is 0.00008060930
Euclidean distance between L and R is 0.00276888666
angle between $W_e^u(p)$ and iterated arc LR = 3.14156 radians, angle in degrees is ≈ 180
angle between vectors (p, L) and (p, R) is 1.24158 angle in degrees is ≈ 71
angle between vectors $(p, T^2(L))$ and $(p, T^2(R))$ is 2.76672 radians, angle in degrees is ≈ 159
angle between vectors $(p, T^4(L))$ and $(p, T^4(R))$ is 3.13482 radians angle in degrees is ≈ 180
angle between vectors $(p, T^6(L))$ and $(p, T^6(R))$ is 3.14036 radians angle in degrees is ≈ 180
angle between vectors $(p, T^8(L))$ and $(p, T^8(R))$ is 3.13746 radians angle in degrees is ≈ 180
angle between vectors $(p, T^{10}(L))$ and $(p, T^{10}(R))$ is 3.12554 radians angle in degrees is ≈ 179
angle between vectors $(p, T^{12}(L))$ and $(p, T^{12}(R))$ is 3.10770 radians angle in degrees is ≈ 178
angle between vectors $(p, T^{14}(L))$ and $(p, T^{14}(R))$ is 3.05600 radians angle in degrees is ≈ 175

Enter gap distance as a real exponent of 2 between 3 and 20
last exponent used is 15.03200000
To finish the program enter exponent=0. Chosen exponent = 11.00000000
gap = 2048.0000000 exponent = 11.0000000
iter= 0 dist(L,R) previous to iteration is 3.1244649405582926E-0010
dist between L and R after applying T^{24} is 7.0403463016490730E-0008
dist(L,L1)+dist(L1,M)+dist(M,R1)+dist(R1,R)= 7.0403463017652905E-0008
iter= 12, dist(L,R) previous to iteration is 6.8753381889103818E-0011
dist between L and R after applying T^{24} is 1.6655785942826604E-0007
dist(L,L1)+dist(L1,M)+dist(M,R1)+dist(R1,R)= 1.6655785942826604E-0007
iter= 24, dist(L,R) previous to iteration is 1.6265415954054225E-0010
dist between L and R after applying T^{24} is 4.1674332229362251E-0008
dist(L,L1)+dist(L1,M)+dist(M,R1)+dist(R1,R)= 4.1674332229362251E-0008
iter= 36, dist(L,R) previous to iteration is 4.0697590197929017E-0011
dist between L and R after applying T^{24} is 3.5735658309920103E-0007
dist(L,L1)+dist(L1,M)+dist(M,R1)+dist(R1,R)= 3.5735658309920103E-0007
iter= 48, dist(L,R) previous to iteration is 3.4898103807564854E-0010
dist between L and R after applying T^{24} is 3.1598931331358240E-0006
dist(L,L1)+dist(L1,M)+dist(M,R1)+dist(R1,R)= 3.1598931331363929E-0006
iter= 60, dist(L,R) previous to iteration is 3.0858331376860077E-0009
dist between L and R after applying T^{24} is 8.7084375703600538E-0005
dist(L,L1)+dist(L1,M)+dist(M,R1)+dist(R1,R)= 8.7084375703625518E-0005
iter= 72, dist(L,R) previous to iteration is 8.5043335648104785E-0008
dist between L and R after applying T^{24} is 1.3240893865632102E-0003
dist(L,L1)+dist(L1,M)+dist(M,R1)+dist(R1,R)= 1.3240893866628222E-0003
distance between iterates is too large or curvature is big

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dist between L and R after applying T^{16} is 2.2719494014277907E-0004

dist(L,L1)+dist(L1,M)+dist(M,R1)+dist(R1,R)= 2.2719494016150623E-0004

D dist between iterates continues to be too large or curvature is big

dist between L and R after applying T^4 is 7.0995114872475991E-0006

dist(L,L1)+dist(L1,M)+dist(M,R1)+dist(R1,R)= 7.0995114872475992E-0006

iter= 74, dist(L,R) previous to iteration is 6.9331166869485615E-0009

dist between L and R after applying T^{24} is 1.3422129968050613E-0005

dist(L,L1)+dist(L1,M)+dist(M,R1)+dist(R1,R)= 1.3422129968054511E-0005

$$\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots$$

iter= 624, dist(L,R) previous to iteration is 4.5100485932166933E-0010

dist between L and R after applying T^{10} is 4.4858004094609686E-0007

dist(L,L1)+dist(L1,M)+dist(M,R1)+dist(R1,R)= 4.4858004094609686E-0007

iter= 629, dist(L,R) previous to iteration is 4.3806644611849589E-0010

Sup distance from fixed point p to point $T^{1258}(y_0)$ is 0.00022627982

L1 distance from fixed point p to point $T^{1258}(y_0)$ is 0.01670265394

Euclidean distance from fixed point p to point $T^{1258}(y_0)$ is 0.00141044319

Euclidean distance from fixed point p to point L is 0.00141021923

Euclidean distance from fixed point p to point R is 0.00141066714

Euclidean distance between L and R is 0.00000044858

angle between $W_e^u(p)$ and iterated arc $LR = 0.00003$ radians, angle in degrees is ≈ 0

angle between vectors (p,L) and (p,R) is 0.00002 angle in degrees is ≈ 0

angle between vectors $(p,T^2(L))$ and $(p,T^2(R))$ is 0.00000 radians, angle in degrees is ≈ 0

angle between vectors $(p,T^4(L))$ and $(p,T^4(R))$ is 0.00000 radians angle in degrees is ≈ 0

angle between vectors $(p,T^6(L))$ and $(p,T^6(R))$ is 0.00000 radians angle in degrees is ≈ 0

angle between vectors $(p,T^8(L))$ and $(p,T^8(R))$ is 0.00000 radians angle in degrees is ≈ 0

angle between vectors $(p,T^{10}(L))$ and $(p,T^{10}(R))$ is 0.00000 radians angle in degrees is ≈ 0

angle between vectors $(p,T^{12}(L))$ and $(p,T^{12}(R))$ is 0.00001 radians angle in degrees is ≈ 0

angle between vectors $(p,T^{14}(L))$ and $(p,T^{14}(R))$ is 0.00009 radians angle in degrees is ≈ 0

Enter gap distance as a real exponent of 2 between 3 and 20

last exponent used is 11.00000000

To finish the program enter exponent=0. Chosen exponent = 0.00000000

Press ENTER to finish the program.

References

[Ar]   ARLOT S., *Étude d'un modèle de dynamique des populations,* DEA de Modélisation Stochastique et Statistique, Université Paris Sud XI (2004), p. available at - http://www.di.ens.fr/~~ arlot/ .

[FJASY]   K. FREDGA, M. JAAROLA, R. A. IMS, H. STEEN, NIGEL G. YOCCOZ, *The 'common vole' in Svalbard identified as Microtus epiroticus by chromosome analysis,* Polar Research, Vol 8, Issue 2 (1990), p. 283-290.

[GP]   P. GRASSBERGER, I. PROCACCIA, *Characterization of Strange Attractors,* Physical Review Letters, Vol 50, No 5 (1983), p. 346-349.

[GP2]   P. GRASSBERGER, I. PROCACCIA, *Measuring the strangeness of strange attractors,* Physica D, 9 (1983), p. 189-208.

[GP3]   P. GRASSBERGER, I. PROCACCIA, *Dimensions and entropies of strange attractors from a fluctuating dynamics approach.,* Physica D, 13 (1984), p. 34-54.

[Ims]   R. A. IMS, *Spatial clumping of sexually receptive females induces space sharing among male voles.,* Nature, 335(6190) (1988), p. 541-543.

[Ims2]   R. A. IMS, " *It is a reasonable assumption that sex ratio is 1:1. We know that this for sure at birth (lab studies). ",* Personal communication,

.

[LBY]   X. LAMBIN, V. BRETAGNOLLE, N. G. YOCCOZ, *Vole population cycles in northern and southern Europe: is there a need for different explanations for single pattern?,* The Journal of animal ecology, 335(75(2) (2006), p. 340-349.

[PM]   J. PALIS, W. DE MELO, *Geometrical Theory of Dynamical Systems,* Springer, 1982.

[ER]   J.-P. ECKMANN, D. RUELLE, *Fundamental limitations for estimating dimensions and Lyapunov exponents in dynamical systems,* Physica D, 56 (1992), p. 185-187.

[Sp]   E. H. SPANIER, *Algebraic Topology,* McGraw-Hill, **New York** 1966.

[STB]   J. C. SCHOUTEN, F. TAKENS, C. M. VAN DEN BLEEK, *Maximum-likelihood estimation of the entropy of an attractor,* Phisical Review E, Vol 49 No 1 (1994), p. 126-129.

[RS]   DAVID RUELLE, DENNIS SULLIVAN, *Current flows and diffeomorphisms,* Topology, 14 (1975), p. 319-327.

[Ta]   F. TAKENS, *Invariants Related to Dimensions and Entropy,* Atas do 13° Coloquio Brasileiro de Matematica , Instituto de Matemática Pura e Aplicada (1983), p. Rio de Janeiro-Brazil.

[Ta1]   F. TAKENS, *Detecting strange attractors in turbulence: Dynamical Systems and Turbulence,* Lecture Notes in Mathematics) (1981), **Springer** .vol 898

[TR]   M. THIEL, M. C. ROMANO, *Estimation of dynamical invariants without embedding by recurrence plots,* Chaos , Vol 14 No 2 (2004), p. 234-243.

[YI]   NIGEL G. YOCCOZ, ROLF A. IMS, *Demography of small mammals in cold regions: the importance of environmental variability,* Ecological Bulletins, 47 (1999), p. 133-144.

[YIS]   N. G. YOCCOZ, R. A. IMS, H. STEEN, *Growth and reproduction in islands and mainland populations of the vole Microtus Epiroticus,* Can. J. Zool., 71 (1993), p. 2518-2527.

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